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ON PERFECT POWERS IN k-GENERALIZED PELL SEQUENCE

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Abstract. Let $k \geqslant 2$ and let $(P_n^{(k)})_{n\geqslant 2-k}$ be the k-generalized Pell sequence defined by

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \ldots + P_{n-k}^{(k)}$$

for $n \geqslant 2$ with initial conditions

$$P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \ldots = P_{-1}^{(k)} = P_0^{(k)} = 0, P_1^{(k)} = 1.$$

In this study, we handle the equation $P_n^{(k)} = y^m$ in positive integers n, m, y, k such that $k, y \geqslant 2$, and give an upper bound on n. Also, we will show that the equation $P_n^{(k)} = y^m$ with $2 \leqslant y \leqslant 1000$ has only one solution given by $P_7^{(2)} = 13^2$.

Keywords: Fibonacci and Lucas numbers; exponential Diophantine equation; linear forms in logarithms; Baker's method

MSC 2020: 11B39, 11D61, 11J86

1. Introduction

Let k, r be integers with $k \geqslant 2$ and $r \neq 0$. Let $(G_n^{(k)})_{n\geqslant 2-k}$ be the linear recurrence sequence of order k defined by

$$G_n^{(k)} = rG_{n-1}^{(k)} + G_{n-2}^{(k)} + \ldots + G_{n-k}^{(k)}$$

for $n \geqslant 2$ with the initial conditions $G_{-(k-2)}^{(k)} = G_{-(k-3)}^{(k)} = \ldots = G_{-1}^{(k)} = G_0^{(k)} = 0$ and $G_1^{(k)} = 1$. For $r = 1$, the sequence $(G_n^{(k)})_{n\geqslant 2-k}$ is called k-generalized Fibonacci

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sequence $(F_n^{(k)})_{n\geqslant 2-k}$ (see [6]). For $r = 2$, the sequence $(G_n^{(k)})_{n\geqslant 2-k}$ is called k-generalized Pell sequence $(P_n^{(k)})_{n\geqslant 2-k}$ (see [13]). The terms of these sequences are called k-generalized Fibonacci numbers and k-generalized Pell numbers, respectively. When $k = 2$, we have Fibonacci and Pell sequences $(F_n)_{n\geqslant 0}$ and $(P_n)_{n\geqslant 0}$, respectively.

There has been much interest in the question, when the terms of a linear recurrence sequence are perfect powers. For instance, in [14], Ljunggren showed that for $n \geqslant 2$, P_n is a perfect square precisely for $P_7 = 13^2$ and $P_n = 2x^2$ precisely for $P_2 = 2$. In [9], Cohn solved the same equations for Fibonacci numbers. Later, these problems were extended by Bugeaud, Mignotte and Siksek (see [8]) for Fibonacci numbers and by Pethő (see [16]) for Pell numbers. Pethő [16] and Cohn [10] independently found all perfect powers in the Pell sequence. They proved:

Theorem 1. *The only positive integer solution* (n, y, m) *with* $m \geqslant 2$ *and* $y \geqslant 2$ *of the Diophantine equation* $P_n = y^m$ *is given by* $(n, m, y) = (7, 2, 13)$.

Bugeaud, Mignotte and Siksek (see [8]) solved the Diophantine equation $F_n = y^p$ for $p \geqslant 2$ using modular approach and classical linear forms in logarithms. Lastly, Bravo and Luca handled this problem with $y = 2$, for k-generalized Fibonacci numbers. They showed in [6] that the Diophantine equation $F_n^{(k)} = 2^m$ in positive integers (n, m) has the solutions $(n, m) = (6, 3)$ for $k = 2$ and $(n, m) = (t, t-2)$ for all $2 \leqslant t \leqslant k+1$.

In this paper, we deal with the Diophantine equation

$$P_n^{(k)} = y^m \tag{1}$$

in positive integers n, m with $k, y \geqslant 2$. Our main result is the following.

Theorem 2. *Let* $2 \leqslant y \leqslant 1000$. *Then Diophantine equation* (1) *has only the solution* $(n, m, k, y) = (7, 2, 2, 13)$.

2. Preliminaries

The characteristic polynomial of the sequence $(P_n^{(k)})_{n\geqslant 2-k}$ is

$$\Psi_k(x) = x^k - 2x^{k-1} - \ldots - x - 1. \tag{2}$$

We know from Lemma 1 of [19] that this polynomial has exactly one positive real root located between 2 and 3. We denote the roots of the polynomial in (2) by

$\alpha_1, \alpha_2, \ldots, \alpha_k$. Particuarly, let $\alpha = \alpha_1$ denote the positive real root of $\Psi_k(x)$. The positive real root $\alpha = \alpha(k)$ is called dominant root of $\Psi_k(x)$. The other roots are strictly inside the unit circle. In [5], the Binet- like formula for k-generalized Pell numbers is given by

$$P_n^{(k)} = \sum_{j=1}^{k} \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \alpha_j^n. \tag{3}$$

It was also shown in [5] that the contribution of the roots inside the unit circle to formula (2) is very small, more precisely the approximation

$$|P_n^{(k)} - g_k(\alpha)\alpha^n| < \frac{1}{2} \tag{4}$$

holds for all $n \geqslant 2 - k$, where

$$g_k(z) = \frac{z - 1}{(k+1)z^2 - 3kz + k - 1}. \tag{5}$$

From [3], we can give the inequality, which will be used in the proof of Lemma 8,

$$\left| \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \right| < 1 \tag{6}$$

for $k \geqslant 2$, where α_j's for $j = 1, 2, \ldots, k$ are the roots of the polynomial in (2).

Throughout this paper, α denotes the positive real root of the polynomial given in (2). The following relation between α and $P_n^{(k)}$ given in [5] is valid for all $n \geqslant 1$.

$$\alpha^{n-2} \leqslant P_n^{(k)} \leqslant \alpha^{n-1}. \tag{7}$$

Furthermore, Kılıç in [13] proved that

$$P_n^{(k)} = F_{2n-1} \tag{8}$$

for all $1 \leqslant n \leqslant k + 1$.

Lemma 3 ([5], Lemma 3.2)**.** *Let $k, l \geqslant 2$ be integers. Then:*

(a) *If $k > l$, then $\alpha(k) > \alpha(l)$, where $\alpha(k)$ and $\alpha(l)$ are the values of α relative to k and l, respectively.*
(b) $\varphi^2(1 - \varphi^{-k}) < \alpha < \varphi^2$, *where $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is the golden section.*
(c) $g_k(\varphi^2) = 1/(\varphi + 2)$.
(d) $0.276 < g_k(\alpha) < \frac{1}{2}$.

For solving equation (1), we use linear forms in logarithms and Baker's theory. For this, we give some notations, lemmas and a theorem.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d}(x - \eta^{(i)}) \in \mathbb{Z}[x],$$

where the a_i's are integers with $\gcd(a_0, \ldots, a_n) = 1$ and $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\eta^{(i)}|, 1\})\right) \tag{9}$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b \geqslant 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

We give some properties of the logarithmic height whose proofs can be found in [7]:

$$h(\eta \pm \gamma) \leqslant h(\eta) + h(\gamma) + \log 2, \tag{10}$$

$$h(\eta\gamma^{\pm 1}) \leqslant h(\eta) + h(\gamma), \tag{11}$$

$$h(\eta^m) = |m|h(\eta). \tag{12}$$

Now, from Lemma 6 given in [4], we can deduce the estimation

$$h(g_k(\alpha)) < 5 \log k \quad \text{for } k \geqslant 2, \tag{13}$$

which will be used in the proof of Lemma 8.

We give a theorem deduced from Corollary 2.3 of Matveev [15], which provides a large upper bound for the subscript n in equation (1) (also see Theorem 9.4 in [8]).

Theorem 4. *Assume that $\gamma_1, \gamma_2, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_t$ are rational integers, and $\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1$ is not zero. Then*

$$|\Lambda| > \exp(-1.4 \cdot 30^{t+3} t^{9/2} D^2 (1 + \log D)(1 + \log B) A_1 A_2 \ldots A_t),$$

where $B \geqslant \max\{|b_1|, \ldots, |b_t|\}$, and $A_i \geqslant \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for all $i = 1, \ldots, t$.

In [12], Dujella and Pethő gave a version of the reduction method based on the Baker and Davenport (see [1]). Then, in [2], the authors proved the following lemma, which is an immediate variation of the result due to Dujella and Pethő from [12]. This lemma is based on the theory of continued fractions and will be used to lower the upper bound obtained by Theorem 4 for the subscript n in (1).

Lemma 5. *Let M be a positive integer, let p/q be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\gamma q\|$, where $\|\cdot\|$ denotes the distance from x to the nearest integer. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v, and w with

$$u \leqslant M \quad \text{and} \quad w \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

The following lemma can be found in [11].

Lemma 6. *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a}|x| \quad \text{and} \quad |x| < \frac{a}{1-\mathrm{e}^{-a}}|\mathrm{e}^x - 1|.$$

Finally, we give the following lemma, which can be found in [17].

Lemma 7. *If $m \geqslant 1$, $T \geqslant (4m^2)^m$ and $x/(\log x)^m < T$, then $x < 2^m T(\log T)^m$.*

Before proving our result, we prove the following lemma, which gives an estimate on n in terms of k and y.

Lemma 8. *All solutions (n, m, k, y) of equation (1) satisfy the inequality*

$$n < 6.81 \cdot 10^{12} k^4 (\log k)^2 \log n \cdot \log y. \tag{14}$$

P r o o f. Assume that $P_n^{(k)} = y^m$ with $m, k, y \geqslant 2$. If $1 \leqslant n \leqslant k+1$, then we have $P_n^{(k)} = F_{2n-1} = y^m$ by (8). $F_{2n-1} = y^m$ is not satisfied for any $n \geqslant 1$ by Theorem 1 given in [8]. Then we suppose that $n \geqslant k+2$, which implies that $n \geqslant 4$. Let α be the positive real root of $\Psi_k(x)$ given in (2). Then $2 < \alpha < \varphi^2 < 3$ by Lemma 3 (b). Using (7), we get $\alpha^{n-2} < y^m < \alpha^{n-1}$. Making necessary calculations, we obtain

$$m < (n-1)\frac{\log \alpha}{\log y} \leqslant (n-1)\frac{\log \varphi^2}{\log 2} < 1.4n \tag{15}$$

for $n \geqslant 4$. Now, let us rearrange (1) using inequality (4). Thus, we have

$$|y^m - g_k(\alpha)\alpha^n| < \frac{1}{2}. \tag{16}$$

If we divide both sides of inequality (16) by $g_k(\alpha)\alpha^n$, from Lemma 3, we get

$$\left|\frac{y^m}{\alpha^n g_k(\alpha)} - 1\right| < \frac{1}{2g_k(\alpha)\alpha^n} < \frac{1}{0.552\alpha^n} < \frac{1.82}{\alpha^n}. \tag{17}$$

In order to use Theorem 4, we take

$$(\gamma_1, b_1) := (y, m), \quad (\gamma_2, b_2) := (\alpha, -n), \quad (\gamma_3, b_3) := (g_k(\alpha), -1).$$

The number field containing γ_1, γ_2, and γ_3 is $\mathbb{K} = \mathbb{Q}(\alpha)$, which has degree $D = k$. We show that the number

$$\Lambda_1 := \frac{y^m}{\alpha^n g_k(\alpha)} - 1$$

is nonzero. In contrast to this, assume that $\Lambda_1 = 0$. Then

$$y^m = \alpha^n g_k(\alpha) = \frac{\alpha - 1}{(k+1)\alpha^2 - 3k\alpha + k - 1}\alpha^n.$$

Conjugating the above equality by some automorphism belonging to the Galois group of the splitting field of $\Psi_k(x)$ over $\mathbb{Q}$ and taking absolute values, we get

$$y^m = \left|\frac{\alpha_i - 1}{(k+1)\alpha_i^2 - 3k\alpha_i + k - 1}\alpha_i^n\right|$$

for some $i > 1$. Using (6) and that $|\alpha_i| < 1$, we obtain from the last equality that

$$y^m = \left|\frac{\alpha_i - 1}{(k+1)\alpha_i^2 - 3k\alpha_i + k - 1}\right||\alpha_i|^n < 1,$$

which is impossible since $y \geqslant 2$. Therefore $\Lambda_1 \neq 0$.

Moreover, since $h(y) = \log y$, $h(\gamma_2) = (\log \alpha)/k < (\log 3)/k$ by (9) and $h(g_k(\alpha)) < 5 \log k$ by (13), we can take $A_1 := k \log y$, $A_2 := \log 3$ and $A_3 := 5k \log k$. Also, since $m \leqslant 1.4n$, it follows that $B := 1.4n$. Thus, taking into account inequality (17) and using Theorem 4, we obtain

$$\frac{1.82}{\alpha^n} > |\Lambda_1| > \exp(-Ck^2(1 + \log k)(1 + \log 1.4n)k \log y \cdot \log 3 \cdot 5k \log k)$$

and so

$$n \log \alpha - \log 1.82 < Ck^2 \cdot 3 \log k \cdot 2 \log n \cdot k \log y \cdot \log 3 \cdot 5k \log k,$$

where $C = 1.4 \cdot 30^6 \cdot 3^{9/2}$ and we have used the fact that $1 + \log k < 3 \log k$ for all $k \geqslant 2$ and $1 + \log 1.4n < 2 \log n$ for $n \geqslant 4$. From the last inequality, a quick computation with Mathematica yields

$$n \log \alpha < 4.72 \cdot 10^{12} k^4 (\log k)^2 \cdot \log n \cdot \log y$$

or

$$n < 6.81 \cdot 10^{12} k^4 (\log k)^2 \cdot \log n \cdot \log y.$$

Thus, the proof is completed. $\square$

3. The proof of Theorem 2

Assume that Diophantine equation (1) is satisfied for $2 \leqslant y \leqslant 1000$. If $1 \leqslant n \leqslant k+1$, then we have $P_n^{(k)} = F_{2n-1} = y^m$ by (8). The equation $F_{2n-1} = y^m$ has no solutions by Theorem 1 given in [8]. Then we suppose that $n \geqslant k+2$. If $k = 2$, then we have $P_n^{(2)} = P_n = y^m$, which implies that $(n, m, k, y) = (7, 2, 2, 13)$ by Theorem 1. Now, assume that $k \geqslant 3$. So, $n \geqslant 5$. On the other hand, since $y \leqslant 1000$, it follows that

$$\frac{n}{\log n} < 4.71 \cdot 10^{13} k^4 (\log k)^2 \tag{18}$$

by (14). By Lemma 7, inequality (18) yields that

$$n < 2T \log T,$$

where $T := 4.71 \cdot 10^{13} k^4 (\log k)^2$. Making necessary calculations, we get

$$n < 3.3 \cdot 10^{15} k^4 (\log k)^3 \tag{19}$$

for all $k \geqslant 3$.

Let $k \in [3, 555]$. Then, we obtain $n < 7.9 \cdot 10^{28}$ from (19). Now, let us try to reduce this upper bound on n by applying Lemma 5. Let

$$z_1 := m \log y - n \log \alpha + \log \frac{1}{g_k(\alpha)}$$

and $x := \mathrm{e}^{z_1} - 1$. Then from (17), it is seen that

$$|x| = |\mathrm{e}^{z_1} - 1| < \frac{1.82}{\alpha^n} < 0.12$$

for $n \geqslant 5$. Choosing $a := 0.12$, we get the inequality

$$|z_1| = |\log(x+1)| < \frac{\log \frac{100}{88}}{0.12} \frac{1.82}{\alpha^n} < \frac{1.94}{\alpha^n}$$

by Lemma 6. Thus, it follows that

$$0 < \left| m \log y - n \log \alpha + \log \frac{1}{g_k(\alpha)} \right| < \frac{1.94}{\alpha^n}.$$

Dividing this inequality by $\log \alpha$, we get

$$0 < |m\gamma - n + \mu| < AB^{-w}, \tag{20}$$

where

$$\gamma := \frac{\log y}{\log \alpha}, \quad \mu := \frac{1}{\log \alpha} \log \frac{1}{g_k(\alpha)}, \quad A := 2.8, \quad B := \alpha, \quad \text{and} \quad w := n.$$

It can be easily seen that $\log y/\log\alpha$ is irrational. If it were not, then we could write $\log y/\log\alpha = b/a$ for some positive integers a and b. This implies that $y^a = \alpha^b$. Conjugating this equality by an automorphism belonging to the Galois group of the splitting field of $\Psi_k(x)$ over $\mathbb{Q}$ and taking absolute values, we get $y^a = |\alpha_i|^b$ for some $i > 1$. This is impossible since $|\alpha_i| < 1$ and $y \geqslant 2$. Put

$$M := 1.106 \cdot 10^{29},$$

which is an upper bound on m since $m < 1.4n < 1.106 \cdot 10^{29}$. Thus, we find that q_{91}, the denominator of the 91th convergent of γ, exceeds $6M$. Furthermore, a quick computation with Mathematica gives us that the value

$$\frac{\log(Aq_{91}/\varepsilon)}{\log B}$$

is less than 164.9 for all $k \in [3, 555]$. So, if (20) has a solution, then

$$n < \frac{\log(Aq_{91}/\varepsilon)}{\log B} \leqslant 164.9,$$

that is, $n \leqslant 164$. In this case, $m < 229$ by (15). A quick computation with Mathematica gives us that the equation $P_n^{(k)} = y^m$ has no solutions for $n \in [5, 164]$, $m \in [2, 229)$ and $k \in [3, 555]$. Thus, this completes the analysis in the case $k \in [3, 555]$.

From now on, we can assume that $k > 555$. Then we can see from (19) that the inequality

$$n < 3.3 \cdot 10^{15} k^4 (\log k)^3 < \varphi^{k/2-2} < \varphi^{k/2} \tag{21}$$

holds for $k > 555$.

Now, let $\lambda > 0$ be such that $\alpha + \lambda = \varphi^2$. By Lemma 3 (b), we obtain

$$\lambda = \varphi^2 - \alpha < \varphi^2 - \varphi^2(1 - \varphi^{-k}) = \varphi^{2-k},$$

i.e.,

$$\lambda < \frac{1}{\varphi^{k-2}}. \tag{22}$$

On the other hand,

$$\begin{aligned}\alpha^n &= (\varphi^2 - \lambda)^n = \varphi^{2n}\Big(1 - \frac{\lambda}{\varphi^2}\Big)^n = \varphi^{2n} \mathrm{e}^{n\log(1-\lambda/\varphi^2)} \\ &\geqslant \varphi^{2n}\mathrm{e}^{-n\lambda} \geqslant \varphi^{2n}(1 - n\lambda) > \varphi^{2n}\Big(1 - \frac{n}{\varphi^{k-2}}\Big),\end{aligned}$$

where we have used the facts that $\log(1 - x) \geqslant -\varphi^2 x$ for $0 < x < 0.907$ and $\mathrm{e}^{-x} > 1 - x$ for all $x \in \mathbb{R} \setminus \{0\}$. Thus,

$$\alpha^n > \varphi^{2n} - \frac{n\varphi^{2n}}{\varphi^{k-2}} > \varphi^{2n} - \frac{\varphi^{2n}}{\varphi^{k/2}}$$

by (21). Since $\alpha < \varphi^2$, it follows that

$$\alpha^n < \varphi^{2n} + \frac{\varphi^{2n}}{\varphi^{k/2}}$$

and so we have

$$|\alpha^n - \varphi^{2n}| < \frac{\varphi^{2n}}{\varphi^{k/2}}. \tag{23}$$

Thus, we can write $\alpha^n = \varphi^{2n} + \delta$ with $|\delta| < \varphi^{2n}/\varphi^{k/2}$. Also, the equality

$$g_k(\alpha) = g_k(\varphi^2) + \eta, \quad |\eta| < \frac{4k}{\varphi^k} \tag{24}$$

is given in Lemma 13 of [18]. Since $g_k(\varphi^2) = 1/(\varphi + 2)$ by Lemma 3 (c), it follows that

$$g_k(\alpha) = \frac{1}{\varphi + 2} + \eta.$$

Now we can give the following result.

Lemma 9. *Let $k > 555$ and let α be the dominant root of the polynomial $\Psi_k(x)$. Let us consider $g_k(x)$ defined in (5) as a function of a real variable. Then*

$$g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi + 2} + \frac{\delta}{\varphi + 2} + \eta\varphi^{2n} + \eta\delta, \tag{25}$$

where δ and η are real numbers such that

$$|\delta| < \frac{\varphi^{2n}}{\varphi^{k/2}} \quad \text{and} \quad |\eta| < \frac{4k}{\varphi^k}. \tag{26}$$

So, using (16), (25) and (26), we obtain

$$\begin{aligned} \left|y^m - \frac{\varphi^{2n}}{\varphi + 2}\right| &= \left|(y^m - g_k(\alpha)\alpha^n) + \frac{\delta}{\varphi + 2} + \eta\varphi^{2n} + \eta\delta\right| \\ &\leqslant |y^m - g_k(\alpha)\alpha^n| + \frac{|\delta|}{\varphi + 2} + |\eta|\varphi^{2n} + |\eta||\delta| \\ &< \frac{1}{2} + \frac{\varphi^{2n}}{\varphi^{k/2}(\varphi + 2)} + \frac{4k\varphi^{2n}}{\varphi^k} + \frac{4k\varphi^{2n}}{\varphi^{3k/2}}. \end{aligned} \tag{27}$$

Dividing both sides of the above inequality by $\varphi^{2n}/(\varphi + 2)$, we get

$$\begin{aligned} |y^m\varphi^{-2n}(\varphi + 2) - 1| &< \frac{\varphi + 2}{2\varphi^{2n}} + \frac{1}{\varphi^{k/2}} + \frac{4k(\varphi + 2)}{\varphi^k} + \frac{4k(\varphi + 2)}{\varphi^{3k/2}} \\ &< \frac{0.05}{\varphi^{k/2}} + \frac{1}{\varphi^{k/2}} + \frac{0.005}{\varphi^{k/2}} + \frac{0.005}{\varphi^{k/2}} = \frac{1.06}{\varphi^{k/2}}, \end{aligned} \tag{28}$$

where we have used the facts that $n \geqslant k+2$ and

$$\frac{4k(\varphi+2)}{\varphi^k} < \frac{0.005}{\varphi^{k/2}} \quad \text{and} \quad \frac{4k(\varphi+2)}{\varphi^{3k/2}} < \frac{0.005}{\varphi^{k/2}} \quad \text{for } k > 555.$$

In order to use the result of Theorem 4, we take

$$(\gamma_1, b_1) := (y, m), \quad (\gamma_2, b_2) := (\varphi, -2n), \quad (\gamma_3, b_3) := (\varphi+2, 1).$$

The number field containing γ_1, γ_2, and γ_3 is $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, which has degree $D = 2$. We show that the number

$$\Lambda_1 := y^m \varphi^{-2n}(\varphi+2) - 1$$

is nonzero. In contrast to this, assume that $\Lambda_1 = 0$. Then $y^m(\varphi+2) = \varphi^{2n}$ and conjugating this relation in $\mathbb{Q}(\sqrt{5})$, we get $y^m(\beta+2) = \beta^{2n}$, where $\beta = \frac{1}{2}(1-\sqrt{5}) = \overline{\varphi}$. The left-hand side of the last equality is always greater than 1, while the right-hand side is smaller than 1 because $n \geqslant k+2 > 512$. This is a contradiction. Therefore $\Lambda_1 \neq 0$. Moreover, since

$$h(\gamma_1) = h(y) = \log y, \quad h(\gamma_2) = h(\varphi) \leqslant \frac{\log \varphi}{2}$$

and

$$h(\gamma_3) \leqslant h(\varphi) + h(2) + \log 2 \leqslant \frac{\log \varphi}{2} + \log 4$$

by (11), we can take $A_1 := 2\log y$, $A_2 := \log \varphi$ and $A_3 := \log 16\varphi$. Also, since $m < 1.4n$ by (15), we can take $B := 2n$. Thus, taking into account inequality (28) and using Theorem 4, we obtain

$$(1.06) \cdot \varphi^{-k/2} > |\Lambda_1| > \exp(-C(1+\log 2n) 2\log y \cdot \log \varphi \cdot \log 16\varphi),$$

where $C = 1.4 \cdot 30^6 3^{9/2} 2^2 (1+\log 2)$. This implies that

$$k < 4.2 \cdot 10^{13} \log n, \tag{29}$$

where we have used the fact that $(1+\log 2n) < 2.1 \log n$ for $n \geqslant k+2 > 557$. On the other hand, from (19) we get

$$\log n < \log(3.3 \cdot 10^{15} k^4 (\log k)^3) < 35.8 + 4\log k + 3\log(\log k) < 43 \log k$$

for $k \geqslant 3$. So, from (29) we obtain

$$k < 4.2 \cdot 10^{13} \cdot 43 \log k,$$

which implies that

$$k < 7.1 \cdot 10^{16}. \tag{30}$$

Substituting this bound of k into (19), we get $n < 4.9 \cdot 10^{87}$, which implies that $m < 6.86 \cdot 10^{87}$ by (15).

Now, let

$$z_2 := m \log y - 2n \log \varphi + \log(\varphi + 2)$$

and $x := 1 - e^{z_2}$. Then

$$|x| = |1 - e^{z_2}| < \frac{1.06}{\varphi^{k/2}} < 0.1$$

by (28) since $k > 555$. Choosing $a := 0.1$, we obtain the inequality

$$|z_2| = |\log(x+1)| < \frac{\log \frac{10}{9}}{0.1} \frac{1.06}{\varphi^{k/2}} < \frac{1.12}{\varphi^{k/2}}$$

by Lemma 6. That is,

$$0 < |m \log y - 2n \log \varphi + \log(\varphi + 2)| < \frac{1.12}{\varphi^{k/2}}.$$

Dividing both sides of the above inequality by $\log \varphi$, it is seen that

$$0 < |m\gamma - 2n + \mu| < AB^{-w}, \tag{31}$$

where

$$\gamma := \frac{\log y}{\log \varphi}, \quad \mu := \frac{\log(\varphi+2)}{\log \varphi}, \quad A := 2.33, \quad B := \varphi \quad \text{and} \quad w := \frac{1}{2}k.$$

It is clear that $\log y / \log \varphi$ is irrational. If it were not, then $\log y / \log \varphi = a/b$ for some positive integers a and b with. Thus, we get $y^b = \varphi^a$. Conjugating this equality in $\mathbb{Q}(\sqrt{5})$, we get $y^b = \beta^a$, which is impossible since $\beta^a < 1$, where $\beta = \frac{1}{2}(1 - \sqrt{5}) = \overline{\varphi}$. Besides, if we take $M := 6.86 \cdot 10^{87}$, which is an upper bound on m, we find that q_{212}, the denominator of the 212th convergent of γ, exceeds $6M$. Furthermore, a quick computation with Mathematica gives us that the value

$$\frac{\log(Aq_{212}/\varepsilon)}{\log B}$$

is less than 614.4. So, if (31) has a solution, then

$$\frac{k}{2} < \frac{\log(Aq_{212}/\varepsilon)}{\log B} \leqslant 614.4,$$

that is, $k \leqslant 1228$. Hence, from (19), we get $n < 2.71 \cdot 10^{30}$, which implies that $m < 3.8 \cdot 10^{30}$ by (15). If we apply again Lemma 5 to inequality (31) with $M := 3.8 \cdot 10^{30}$, we find that q_{84}, the denominator of the 84th convergent of γ, exceeds $6M$. After doing this, a quick computation with Mathematica shows that inequality (31) has solutions only for $k \leqslant 552$. This contradicts the fact that $k > 555$. Thus, the proof is completed. $\square$

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