ON THE GENERALIZED B-SCROLLS WITH P th DEGREE IN n-DIMENSIONAL MINKOWSKI SPACES AND STRICTION (CENTRAL) SPACES

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ABSTRACT

In this paper, generalized b-scrolls with p^th degree are introduced in the n-dimensional Minkowski space R_1^n*. Asymptotic bundle and tangential bundle are defined. In the case of space-like or time-like Frenet vectors , the equation of central space is computed.*
Keywords: *B-scroll, time-like, ruled surfaces, central spaces.*

n-BOYUTLU MİNKOWSKİ UZAYINDA P. DERECEDEN GENELLEŞTİRLMİŞ B-SCROLLAR VE STRİKSİYON(MERKEZ) UZAYLAR

ÖZET

Bu çalışmada, n-boyutlu Minkowski uzayında, p.mertebeden b-scrollar tanımlandı. Asimptotik ve teğetsel demetler yardımı ile Frenet vektörlerinin space-like veya time-like olması durumlarında oluşan merkez uzayın denklemi ifade edildi.
Anahtar Kelimeler: *B-scroll, time-like, regle yüzeyler, merkez uzaylar*

1. INTRODUCTION

First of all b-scrolls were introduced in the 3-dimensional Minkowski space R_1^3, [1] and [2]. For an integer q with 0 < q< n , changing the first plus signs above to minus gives a metric tensor

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{q} v^i w^i + \sum_{j=q+1}^{n} v^j w^j$$

of index q. The resulting semi-Euclidean space R_q^n reduces to R^n if q=0. For n> 2, R_1^n is called Minkowski n-space ,[3] . In the n-dimensional Minkowski space R_1^n, lorentz metric is

$$\langle v_p, w_p \rangle = -v^1 w^1 + \sum_{j=2}^{n} v^j w^j$$

In the n-dimensional semi-euclidean space R_q^n, if the Frenet vectors of curve $\eta(I)$ with arc length t are $V_1, V_2, ..., V_r$, the Frenet formulas can be given by the following equations

$$\begin{aligned} \dot{V}_1 &= k_1 V_2 \\ &\vdots \\ \dot{V}_j &= -\varepsilon_{j-2}\varepsilon_{j-1} k_{j-1} V_{j-1} + k_j V_{j+1} \\ &\vdots \\ \dot{V}_r &= -\varepsilon_{r-2}\varepsilon_{r-1} k_{r-1} V_{r-1}. \end{aligned}$$

Here $\varepsilon_{i-1} = \langle V_i, V_i \rangle$ and $i \geq r$ *for* $k_i \neq 0$, [4] and [5].

In the n- dimensional Minkowski space, since the index q is 1 , only one of the $\varepsilon_{i-1} = \langle V_i, V_i \rangle, 1 < i < r,$ will take the value -1 . Here, since $\eta(I)$ is time-like curve, then V_1 is a time-like vector. Hence, only $\varepsilon_0 = -1$. As $V_2, V_3, ..., V_r$ are space-like, then $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = ... = \varepsilon_{r-1} = +1$.

If V_1 is a time-like vector , then the Frenet formulas can be given by the following matrix form ,

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \vdots \\ \dot{V}_{r-2} \\ \dot{V}_{r-1} \\ \dot{V}_r \end{bmatrix} = \begin{bmatrix} 0 & k_2 & 0 & 0 & \cdots & & & 0 \\ k_1 & 0 & k_2 & 0 & \ddots & & & \vdots \\ 0 & -k_2 & 0 & k_3 & \ddots & & & \\ 0 & 0 & -k_3 & 0 & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & 0 & k_{r-2} & 0 \\ & & & & & -k_{r-2} & 0 & k_{r-1} \\ 0 & \cdots & & & & 0 & -k_{r-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \\ V_{r-2} \\ V_{r-1} \\ V_r \end{bmatrix}$$

Similary, if V_2 is a time-like vector, then

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \\ \vdots \\ \dot{V}_{r-2} \\ \dot{V}_{r-1} \\ \dot{V}_r \end{bmatrix} = \begin{bmatrix} 0 & k_2 & 0 & 0 & \cdots & & & 0 \\ k_1 & 0 & k_2 & 0 & \ddots & & & \vdots \\ 0 & k_2 & 0 & k_3 & \ddots & & & \\ 0 & 0 & -k_3 & 0 & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & & 0 & k_{r-2} & 0 \\ & & & & & -k_{r-2} & 0 & k_{r-1} \\ 0 & \cdots & & & & 0 & -k_{r-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \\ V_{r-2} \\ V_{r-1} \\ V_r \end{bmatrix}$$

is the matrix form of the Frenet formulas. Similary, for each time-like vector V_i, matrix form of the Frenet formulas can be obtained.

Definition 1. In the $n-$dimensional Minkowski space R_1^n, $\eta(I)$ is a time-like curve with arc length t. If the Frenet vectors are $\{V_1, V_2, ..., V_r\}$, then

$$Sp\{V_1, V_2, ..., V_p\}; \; p < r < n$$

is the time-like oskulator space with p th degree. In this case,

$$\varphi(t, u_{p+1}, u_{p+2}, ..., u_r) = \eta(t) + \sum_{j=p+1}^{r} u_j V_j(t)$$

is the parametrization of generalized $b-$scroll with p^{th} degree.The directrix of this generalized $b-$scroll with p^{th} degree, is the time-like curve $\eta(I)$. That is $\dot{\eta}(t)=V_1$ is a time-like vector. The space-like generating space of generalized $b-$scroll with p^{th} degree has span with subvectors

$$\{V_{p+1}, V_{p+2}, ..., V_r\}.$$

Since this generating space is $(r-p)-$dimensional , it can be shown by E_{r-p} . The dimension of this special surface b-scroll is $(r-p)+1$.

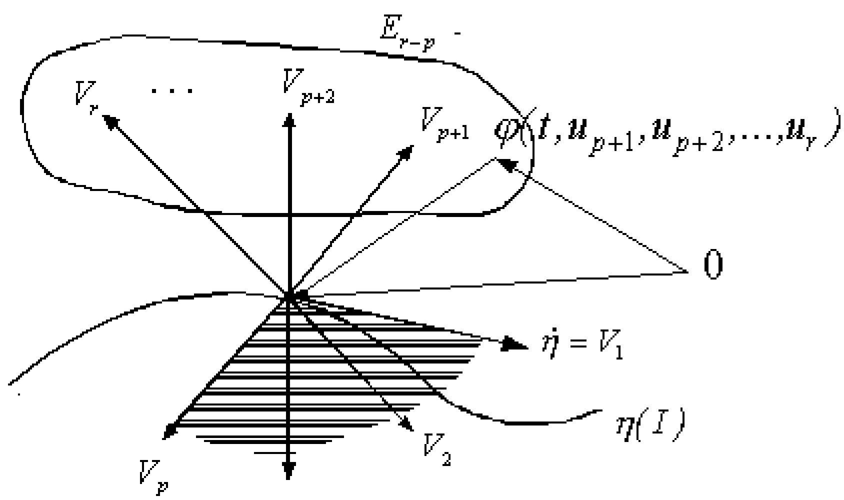

Figure 1: The generalized b-scrolls with p^{th} degree.

Let M be this surface whose ordered basis tangent vectors at the point $\eta(t)$ are given as follows:

$$\begin{aligned}
\varphi_t &= \dot{\eta}(t) + \sum_{j=p+1}^{r} u_j \dot{V}_j(t) = V_1 + \sum_{j=p+1}^{r} u_j \dot{V}_j(t) \\
\varphi_{u_{p+1}} &= V_{p+1} \\
\varphi_{u_{p+2}} &= V_{p+2} \\
&\vdots \\
\varphi_{u_r} &= V_r .
\end{aligned}$$

Definition 2. In the $n-$dimensional Minkowski space R_1^n, the asymptotic bundle, [6], of generalized $b-$scroll with p^{th} degree, is denoted by

$$A(t) = Sp\{V_{p+1}, V_{p+2},..., V_r, \dot{V}_{p+1}, \dot{V}_{p+2},..., \dot{V}_r\}.$$

Since

$$\begin{aligned}
\dot{V}_{p+1} &= -k_p V_p + k_{p+1} V_{p+2} \\
\dot{V}_{p+2} &= -k_{p+1} V_{p+1} + k_{p+2} V_{p+3} \\
&\vdots
\end{aligned}$$

Then only the vector $\dot{V}_{p+1}$ is linearly independent from vectors $V_{p+1}, V_{p+2},..., V_r$. On the other hand, the vectors $\dot{V}_{p+2},..., \dot{V}_r$ are dependent on the vectors $V_{p+1}, V_{p+2},..., V_r$. All these vectors are space-like vectors.

$$\{V_p, V_{p+1}, V_{p+2},..., V_r\}$$

is an orthonormal basis of $A(t)$ and dim $A(t) = r - p + 1$. The asymptotic bundle $A(t)$ is space-like because, unique time-like vector V_1 of Frenet vectors is not an element of $A(t)$.

Definition 3. In the $n-$dimensional Minkowski space R_1^n, denote the tangential bundle ,[6], of the generalized $b-$scroll with p^{th} degree, by

$$T(t) = Sp\{V_{p+1}, V_{p+2},..., V_r, \dot{V}_{p+1}, \dot{V}_{p+2},..., \dot{V}_r, \dot{\eta}\}$$

Since

$$\begin{aligned} \dot{V}_{p+1} &= -k_p V_p + k_{p+1} V_{p+2} \\ \dot{V}_{p+2} &= -k_{p+1} V_{p+1} + k_{p+2} V_{p+3} \\ &\vdots \end{aligned}$$

only the two vectors $\dot{\eta} = V_1$ and $\dot{V}_{p+1}$ are independent from vectors $V_{p+1}, V_{p+2}, ..., V_r$. The vectors $\dot{V}_{p+2}, ..., \dot{V}_r$ are dependent on the vectors $V_{p+1}, V_{p+2}, ..., V_r$. The vectors $V_{p+1}, V_{p+2}, ..., V_r$ are space-like, but $\dot{\eta} = V_1$ is time-like.

$$\{V_1, V_p, V_{p+1}, V_{p+2}, ..., V_r\}$$

is the orthonormal basis vectors of $T(t)$ and dim $T(t) = r - p + 2$. $T(t)$ is time-like because, the time-like vector V_1 is an element of $T(t)$.

Definition 4. In the $n-$dimensional Minkowski space R_1^n, since dim $A(t) \neq$ dim $T(t)$, the generalized $b-$scroll with p^{th} degree and with time-like directrix, has not an edge space but , there is a striction (central) space, [6]. The dimension of this striction (central) space is $(r-p)$. Vectors V_{p+i}, $for\ 1 < i,$ are space-like thus we can calculate the striction space as in the Euclidean space. That is, the position vectors of the striction space are the solutions of the differantial equation system , which has the following matrix form:

$$\begin{bmatrix} \dot{u}_{p+1} \\ \dot{u}_{p+2} \\ \dot{u}_{p+3} \\ \vdots \\ \dot{u}_{r-2} \\ \dot{u}_{r-1} \\ \dot{u}_r \end{bmatrix} = \begin{bmatrix} 0 & k_{p+1} & 0 & \cdots & & & 0 \\ -k_{p+1} & 0 & k_{p+2} & \ddots & & & \vdots \\ 0 & -k_{p+2} & 0 & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & & & \\ & & & & 0 & k_{r-2} & 0 \\ & & & & -k_{r-2} & 0 & k_{r-1} \\ 0 & \cdots & & & 0 & -k_{r-1} & 0 \end{bmatrix} \begin{bmatrix} u_{p+1} \\ u_{p+2} \\ u_{p+3} \\ \vdots \\ u_{r-2} \\ u_{r-1} \\ u_r \end{bmatrix} \quad (1).$$

Corollary: In the $n-$dimensional Minkowski space R_1^n , if one of the vectors

$V_1, V_2, V_3, ..., V_p$ is time-like, the position vectors of the striction space of the generalized $b-$scroll with p^{th} degree will be the same with the solutions of the equation system which has the matrix form given above (1) .

Definition 5. In the $n-$dimensional Minkowski space R_1^n, $\eta(I)$ is a space-like curve with arc length t. If $\{V_1, V_2, ..., V_r\}$ are the Frenet vectors, then

$$Sp\{V_1, V_2, ..., V_p\};\ p < r <, n$$

is the space-like osculator space with p^th degree. In this case,

$$\varphi(t, u_{p+1}, u_{p+2}, ..., u_r) = \eta(t) + \sum_{j=p+1}^{r} u_j V_j(t)$$

is the parametrization of generalized $b-$scroll with p^{th} degree.The directrix of this generalized $b-$scroll with p^{th} degree, is the space-like curve $\eta(I)$, that is $\dot{\eta}(t) = V_1$ a space-like vector .

$$E_{r-p} = Sp\{V_{p+1}, V_{p+2}, ..., V_r\}$$

is the time-like generating space of the generalized $b-$scroll with p^{th} degree. Only one of the vectors $V_{p+1}, V_{p+2}, ..., V_r$ is a time-like vector, since the index q is 1.

First of all, let V_{p+1} be a time-like vector. It means that $\varepsilon_p = \langle V_{p+1}, V_{p+1} \rangle = -1$ and $\varepsilon_{p+1} = \langle V_{p+2}, V_{p+2} \rangle, ..., \varepsilon_{r-1} = \langle V_r, V_r \rangle = 1$.

According to the definitions of the asymptotic bundle and the tangential bundle of generalized $b-$scroll with p^{th} degree,

$$
\begin{aligned}
\dot{V}_p &= -\varepsilon_{p-2}\varepsilon_{p-1}k_{p-1}V_{p-1} + k_pV_{p+1} \\
\dot{V}_{p+1} &= -\varepsilon_{p-1}\varepsilon_p k_p V_p + k_{p+1}V_{p+2} \\
&= k_pV_p + k_{p+1}V_{p+2} \\
\dot{V}_{p+2} &= -\varepsilon_p\varepsilon_{p+1}k_{p+1}V_{p+1} + k_{p+2}V_{p+3} \\
&= k_{p+1}V_{p+1} + k_{p+2}V_{p+3} \\
\dot{V}_{p+3} &= -\varepsilon_{p+1}\varepsilon_{p+2}k_{p+2}V_{p+2} + k_{p+3}V_{p+4} \\
&= -k_{p+2}V_{p+2} + k_{p+3}V_{p+4} \\
&\vdots
\end{aligned}
$$

are obtained by using Frenet formulas. If V_{p+1} is time-like , then only first terms of vectors $\dot{V}_{p+1}$ and $\dot{V}_{p+2}$ will change their signs. However, other signs will not change.

$p(t)$ is any curve family with equation

$$p(t) = \eta(t) + \sum_{j=p+1}^{r} u_j(t)V_j(t)$$

and it has the derivative

$$
\begin{aligned}
\dot{p}(t) &= \dot{\eta} + \sum_{j=p+1}^{r} \dot{u}_j V_j + \sum_{j=p+1}^{r} u_j \dot{V}_j \\
&= V_1 + \sum_{j=p+1}^{r} \dot{u}_j V_j + \sum_{j=p+1}^{r-1} u_j \left(-\varepsilon_{j-2}\varepsilon_{j-1}k_{j-1}V_{j-1} + k_jV_{j+1}\right) - \varepsilon_{r-2}\varepsilon_{r-1}u_rk_{r-1}V_{r-1} \\
&= V_1 + \sum_{j=p+1}^{r} \dot{u}_j V_j - \varepsilon_{j-2}\varepsilon_{j-1} \sum_{j=p+1}^{r-1} u_jk_{j-1}V_{j-1} + \sum_{j=p+1}^{r-1} u_jk_jV_{j+1} - \varepsilon_{r-2}\varepsilon_{r-1}u_rk_{r-1}V_{r-1} \\
&= V_1 + \dot{u}_{p+1}V_{p+1} + \dot{u}_{p+2}V_{p+2} + \dot{u}_{p+3}V_{p+3} + \ldots + \dot{u}_{r-2}V_{r-2} + \dot{u}_{r-1}V_{r-1} + \dot{u}_rV_r \\
&\quad + u_{p+1}k_pV_p + u_{p+2}k_{p+1}V_{p+1} - u_{p+3}k_{p+2}V_{p+2} - u_{p+4}k_{p+3}V_{p+3} - \ldots \\
&\quad - u_{r-2}k_{r-3}V_{r-3} - u_{r-1}k_{r-2}V_{r-2} + u_{p+1}k_{p+1}V_{p+2} + u_{p+2}k_{p+2}V_{p+3} + \ldots \\
&\quad + u_{r-3}k_{r-3}V_{r-2} + u_{r-2}k_{r-2}V_{r-1} + u_{r-1}k_{r-1}V_r - u_rk_{r-1}V_{r-1} \\
&= V_1 + u_{p+1}k_pV_p + \left(\dot{u}_{p+1} + u_{p+2}k_{p+1}\right)V_{p+1} + \left(\dot{u}_{p+2} + u_{p+1}k_{p+1} - u_{p+3}k_{p+2}\right)V_{p+2} \\
&\quad + \left(\dot{u}_{p+3} + u_{p+2}k_{p+2} - u_{p+4}k_{p+3}\right)V_{p+3} + \ldots + \left(\dot{u}_{r-2} + u_{r-3}k_{r-3} - u_{r-1}k_{r-2}\right)V_{r-2} \\
&\quad + \left(\dot{u}_{r-1} + u_{r-2}k_{r-2} - u_rk_{r-1}\right)V_{r-1} + \left(\dot{u}_r + u_{r-1}k_{r-1}\right)V_r .
\end{aligned}
$$

If there exist a common perpendicular to two constructive rullings in the skew surface , then the foot of common perpendicular on the main rulling is called the central point. The locus of central points is called the striction space,[7].

Under the condition of orthonormalizm , the solution vectors u of the equation

$$\left\langle \dot{p}(t), \frac{d}{dt}\left[\sum_{i=p+1}^{r} u_i(t)V_i(t)\right]\right\rangle = 0$$

are the position vectors of the striction space. This equation implies that

$$\begin{aligned}&\left(u_{p+1}k_p\right)^2 - \left(\dot{u}_{p+1} + u_{p+2}k_{p+1}\right)^2 + \left(\dot{u}_{p+2} + u_{p+1}k_{p+1} - u_{p+3}k_{p+2}\right)^2 \\ &+ \left(\dot{u}_{p+3} + u_{p+2}k_{p+2} - u_{p+4}k_{p+3}\right)^2 + \ldots + \left(\dot{u}_{r-2} + u_{r-3}k_{r-3} - u_{r-1}k_{r-2}\right)^2 \\ &+ \left(\dot{u}_{r-1} + u_{r-2}k_{r-2} - u_r k_{r-1}\right)^2 + \left(\dot{u}_r + u_{r-1}k_{r-1}\right)^2 = 0.\end{aligned}$$

If $u_{p+1}k_p = 0$, then, $u_{p+1} \neq 0$ then $k_p = 0$ or if $k_p \neq 0$ and $u_{p+1} = 0$. In the other terms, we can continue on the similiar way. Let assume that all of the curvatures k_i be different from zero. In this condition , if $u_{p+1} = 0,$ we can take $\dot{u}_{p+1} = 0$, $u_{p+2} = 0 \Rightarrow \dot{u}_{p+2} = 0 \Rightarrow u_{p+3} = 0 \Rightarrow \ldots$ So, the space-like directrix $\eta(I)$ of this generalized $b-$scroll with p[th] degree, is the striction space. Under the special condition

$$\dot{u}_{p+1} + u_{p+2}k_{p+1} = 0$$

we can solve the differential equation system. Using the equations

$$\begin{aligned}\dot{u}_{p+1} &= -k_{p+1}u_{p+2} \\ \dot{u}_{p+2} &= k_{p+2}u_{p+3} - k_{p+1}u_{p+1} \\ \dot{u}_{p+3} &= k_{p+3}u_{p+4} - k_{p+2}u_{p+2} \\ &\vdots \\ \dot{u}_{r-2} &= k_{r-2}u_{r-1} - k_{r-3}u_{r-3} \\ \dot{u}_{r-1} &= k_{r-1}u_r - k_{r-2}u_{r-2} \\ \dot{u}_r &= -k_{r-1}u_{r-1}\end{aligned}$$

we can obtain Lyapunov matrix

$$\begin{bmatrix} \dot{u}_{p+1} \\ \dot{u}_{p+2} \\ \dot{u}_{p+3} \\ \vdots \\ \dot{u}_{r-1} \\ \dot{u}_{r} \end{bmatrix} = \begin{bmatrix} 0 & -k_{p+1} & 0 & \cdots & & 0 \\ -k_{p+1} & 0 & k_{p+2} & \ddots & & \vdots \\ 0 & -k_{p+2} & 0 & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ & & & & 0 & k_{r-1} \\ 0 & \cdots & & & -k_{r-1} & 0 \end{bmatrix} \begin{bmatrix} u_{p+1} \\ u_{p+2} \\ u_{p+3} \\ \vdots \\ u_{r-1} \\ u_{r} \end{bmatrix}.$$

That is, the position vectors of the striction space are the solutions of the homogeneous differantial equation

$$\dot{U}(t) = A(t)U(t).$$

In further studies, it is possible to seek for other solutions , except these special solutions.

Now let V_{p+2} be a time-like vector , in the time-like generating space E_{r-p} of generalized $b-$scroll with p^{th} degree. It means that

$$\varepsilon_{p+1} = \langle V_{p+2}, V_{p+2} \rangle = -1$$

and

$$\varepsilon_{p} = \langle V_{p+1}, V_{p+1} \rangle = 1, \varepsilon_{p+2} = \langle V_{p+3}, V_{p+3} \rangle = 1, ..., \varepsilon_{r-1} = \langle V_{r}, V_{r} \rangle = 1$$

are obtained. According to the definitions of the asymptotic bundle and the tangential bundle of generalized $b-$scroll with p^{th} degree ,

$$
\begin{aligned}
\dot{V}_{p+1} &= -\varepsilon_{p-1}\varepsilon_p k_p V_p + k_{p+1}V_{p+2} \\
&= -k_p V_p + k_{p+1}V_{p+2} \\
\dot{V}_{p+2} &= -\varepsilon_p\varepsilon_{p+1}k_{p+1}V_{p+1} + k_{p+2}V_{p+3} \\
&= k_{p+1}V_{p+1} + k_{p+2}V_{p+3} \\
\dot{V}_{p+3} &= -\varepsilon_{p+1}\varepsilon_{p+2}k_{p+2}V_{p+2} + k_{p+3}V_{p+4} \\
&= k_{p+2}V_{p+2} + k_{p+3}V_{p+4} \\
\dot{V}_{p+4} &= -\varepsilon_{p+2}\varepsilon_{p+3}k_{p+3}V_{p+3} + k_{p+4}V_{p+5} \\
&= -k_{p+3}V_{p+3} + k_{p+4}V_{p+5} \\
&\vdots
\end{aligned}
$$

are obtained by using Frenet formulas. It is obvious that, if V_{p+2} is time-like , then only the first terms of vectors $\dot{V}_{p+2}$ and $\dot{V}_{p+3}$ will change their signatures. The others will not change.

$p(t)$ is any curve family with equation

$$p(t) = \eta(t) + \sum_{j=p+1}^{r} u_j(t)V_j(t)$$

and it has the differantial form

$$
\begin{aligned}
\dot{p}(t) = {} & V_1 - u_{p+1}k_pV_p + \;\; \dot{u}_{p+1} + u_{p+2}k_{p+1} \;\; V_{p+1} + \;\; \dot{u}_{p+2} + u_{p+1}k_{p+1} + u_{p+3}k_{p+2} \;\; V_{p+2} \\
& + \;\; \dot{u}_{p+3} + u_{p+2}k_{p+2} - u_{p+4}k_{p+3} \;\; V_{p+3} + \ldots + \;\; \dot{u}_{r-2} + u_{r-3}k_{r-3} - u_{r-1}k_{r-2} \;\; V_{r-2} \\
& + \;\; \dot{u}_{r-1} + u_{r-2}k_{r-2} - u_r k_{r-1} \;\; V_{r-1} + \;\; \dot{u}_r + u_{r-1}k_{r-1} \;\; V_r .
\end{aligned}
$$

Under the condition of orthonormalism, the solution vectors u of the equation

$$\left\langle \dot{p}(t), \frac{d}{dt}\left[\sum_{i=p+1}^{r} u_i(t)V_i(t)\right]\right\rangle = 0$$

are the position vectors of the striction curve (space). This equation implies that

$$
\begin{aligned}
&-\left(u_{p+1}k_p\right)^2+\left(\dot u_{p+1}+u_{p+2}k_{p+1}\right)^2-\left(\dot u_{p+2}+u_{p+1}k_{p+1}+u_{p+3}k_{p+2}\right)^2\\
&+\left(\dot u_{p+3}+u_{p+2}k_{p+2}-u_{p+4}k_{p+3}\right)^2+\ldots+\left(\dot u_{r-2}+u_{r-3}k_{r-3}-u_{r-1}k_{r-2}\right)^2\\
&+\left(\dot u_{r-1}+u_{r-2}k_{r-2}-u_r k_{r-1}\right)^2+\left(\dot u_r+u_{r-1}k_{r-1}\right)^2=0
\end{aligned}
$$

If $u_{p+1}k_p = 0$, then $u_{p+1} \neq 0$ then, $k_p = 0$ or if $k_p \neq 0$ and $u_{p+1} = 0$. In the other terms we can continue on the similiar way. Let assume that all of the curvatures k_i be different from zero. In this condition , if $u_{p+1} = 0$, we can take $\dot u_{p+1} = 0$, $u_{p+2} = 0 \Rightarrow \dot u_{p+2} = 0 \Rightarrow u_{p+3} = 0 \Rightarrow \ldots$ So, the space-like directrix $\eta(I)$ of this generalized $b-$scroll with p th degree, is the striction space.

Under the special condition

$$\dot u_{p+2} + u_{p+1}k_{p+1} + u_{p+3}k_{p+2} = 0$$

we can solve the differential equation system. Using the equations

$$
\begin{aligned}
\dot u_{p+1} &= -k_{p+1}u_{p+2}\\
\dot u_{p+2} &= -k_{p+1}u_{p+1} - k_{p+2}u_{p+3}\\
\dot u_{p+3} &= k_{p+3}u_{p+4} - k_{p+2}u_{p+2}\\
&\vdots\\
\dot u_{r-2} &= k_{r-2}u_{r-1} - k_{r-3}u_{r-3}\\
\dot u_{r-1} &= k_{r-1}u_r - k_{r-2}u_{r-2}\\
\dot u_r &= -k_{r-1}u_{r-1}
\end{aligned}
\quad ,
$$

we can obtain Lyapunov matrix

$$
\begin{bmatrix}\dot u_{p+1}\\ \dot u_{p+2}\\ \dot u_{p+3}\\ \vdots\\ \dot u_{r-2}\\ \dot u_{r-1}\\ \dot u_r\end{bmatrix}
=
\begin{bmatrix}
0 & -k_{p+1} & 0 & 0 & \cdots & & 0\\
-k_{p+1} & 0 & -k_{p+2} & 0 & \ddots & & \vdots\\
0 & -k_{p+2} & 0 & k_{p+3} & \ddots & & \\
\vdots & \ddots & -k_{p+3} & \ddots & \ddots & & \\
 & & & \ddots & & k_{r-2} & 0\\
 & & & & -k_{r-2} & 0 & k_{r-1}\\
0 & \cdots & & & 0 & -k_{r-1} & 0
\end{bmatrix}
\begin{bmatrix}u_{p+1}\\ u_{p+2}\\ u_{p+3}\\ \vdots\\ u_{r-2}\\ u_{r-1}\\ u_r\end{bmatrix}
$$

That is, the position vectors of the striction space are the solutions of the homogeneous differantial equation

$$\dot{U}(t) = A(t)U(t).$$

In further studies, it is possible to seek for the other solutions, except these special solutions.

Finally, let V_r be the time-like vector of the time-like generating space E_{r-p} of generalized $b-$scroll with p th degree. It means that

$$\varepsilon_{r-1} = \langle V_r, V_r \rangle = -1$$

and

$$\varepsilon_p = \langle V_{p+1}, V_{p+1} \rangle = 1, \varepsilon_{p+1} = \langle V_{p+2}, V_{p+2} \rangle = 1, ..., \varepsilon_{r-2} = \langle V_{r-1}, V_{r-1} \rangle = 1$$

are obtained. According to the definitions of the asymptotic bundle and the tangential bundle of generalized $b-$scroll with p th degree,

$$\begin{aligned} \dot{V}_{r-1} &= -\varepsilon_{r-3}\varepsilon_{r-2}k_{r-2}V_{r-2} + k_{r-1}V_r \\ &= -k_{r-2}V_{r-2} + k_{r-1}V_r \\ \dot{V}_r &= -\varepsilon_{r-2}\varepsilon_{r-1}k_{r-1}V_{r-1} \\ &= k_{r-1}V_{r-1} \end{aligned}$$

are obtained by using Frenet formulas. It is obvious that, if V_r is time-like , then only $\dot{V}_r$ will change its signature.The others will not change.

$p(t)$ is any curve family with equation

$$p(t) = \eta(t) + \sum_{j=p+1}^{r} u_j(t)V_j(t)$$

and it has the differential form

$$\begin{aligned} \dot{p}(t) = {} & V_1 - u_{p+1}k_pV_p + (\dot{u}_{p+1} - u_{p+2}k_{p+1})V_{p+1} + (\dot{u}_{p+2} + u_{p+1}k_{p+1} - u_{p+3}k_{p+2})V_{p+2} \\ & + (\dot{u}_{p+3} + u_{p+2}k_{p+2} - u_{p+4}k_{p+3})V_{p+3} + ... + (\dot{u}_{r-2} + u_{r-3}k_{r-3} - u_{r-1}k_{r-2})V_{r-2} \\ & + (\dot{u}_{r-1} + u_{r-2}k_{r-2} + u_rk_{r-1})V_{r-1} + (\dot{u}_r + u_{r-1}k_{r-1})V_r. \end{aligned}$$

Under the condition of orthonormalism, the solution vectors u of the equation

$$\left\langle \dot{p}(t), \frac{d}{dt}\left[\sum_{i=p+1}^{r} u_i(t)V_i(t)\right]\right\rangle = 0$$

are the position vectors of the striction curve (space).This equation implies that

$$\begin{aligned}
&\left(u_{p+1}k_p\right)^2 + \left(u_{p+1} - u_{p+2}k_{p+1}\right)^2 + \left(u_{p+2} + u_{p+1}k_{p+1} - u_{p+3}k_{p+2}\right)^2 \\
&+ \left(u_{p+3} + u_{p+2}k_{p+2} - u_{p+4}k_{p+3}\right)^2 + \ldots + \left(u_{r-2} + u_{r-3}k_{r-3} - u_{r-1}k_{r-2}\right)^2 \\
&+ \left(u_{r-1} + u_{r-2}k_{r-2} + u_r k_{r-1}\right)^2 - \left(u_r + u_{r-1}k_{r-1}\right)^2 = 0
\end{aligned}$$

If $u_{p+1}k_p = 0$, then $u_{p+1} \neq 0$ then $k_p = 0$ or if $k_p \neq 0$ and $u_{p+1} = 0$. In the other terms we can continue on the similiar way. Let assume that all of the curvatures k_i be different from zero. In this condition , if $u_{p+1} = 0$, we can take $\dot{u}_{p+1} = 0$, $u_{p+2} = 0 \Rightarrow \dot{u}_{p+2} = 0 \Rightarrow u_{p+3} = 0 \Rightarrow \ldots$ So, the space-like directrix $\eta(I)$ of this generalized $b-$scroll with p th degree, is the striction space. Under the special condition.

$$\left(u_r + u_{r-1}k_{r-1}\right) = 0$$

we can solve the differential equation system.Using the equations

$$\begin{aligned}
\dot{u}_{p+1} &= k_{p+1}u_{p+2} \\
\dot{u}_{p+2} &= -k_{p+1}u_{p+1} + k_{p+2}u_{p+3} \\
\dot{u}_{p+3} &= -k_{p+2}u_{p+2} + k_{p+3}u_{p+4} \\
&\vdots \\
\dot{u}_{r-2} &= -k_{r-3}u_{r-3} + k_{r-2}u_{r-1} \\
\dot{u}_{r-1} &= -k_{r-1}u_r - k_{r-2}u_{r-2} \\
\dot{u}_r &= -k_{r-1}u_{r-1}
\end{aligned}$$

we can obtain Lyapunov matrix

$$\begin{bmatrix} \dot{u}_{p+1} \\ \dot{u}_{p+2} \\ \dot{u}_{p+3} \\ \vdots \\ \dot{u}_{r-2} \\ \dot{u}_{r-1} \\ \dot{u}_{r} \end{bmatrix} = \begin{bmatrix} 0 & k_{p+1} & 0 & 0 & \cdots & & 0 \\ -k_{p+1} & 0 & k_{p+2} & 0 & \ddots & & \vdots \\ 0 & -k_{p+2} & 0 & k_{p+3} & \ddots & & \\ \vdots & \ddots & -k_{p+3} & \ddots & \ddots & & \\ & & & \ddots & & k_{r-2} & 0 \\ & & & & -k_{r-2} & 0 & -k_{r-1} \\ 0 & \cdots & & & 0 & -k_{r-1} & 0 \end{bmatrix} \begin{bmatrix} u_{p+1} \\ u_{p+2} \\ u_{p+3} \\ \vdots \\ u_{r-2} \\ u_{r-1} \\ u_{r} \end{bmatrix}.$$

That is, the position vectors of the striction space are the solutions of the homogeneous differantial equation

$$\dot{U}(t) = A(t)U(t).$$

In further studies, it is possible to seek for the other solutions, except these special solutions.

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