The Accuracy of The Finite Element Method For Non-Reflecting Boundaries

Yansıtmıyan Sınırların Kullanılışında Sonlu Eleman Yönteminin Doğruluğu

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SUMMARY

The limitations on the size of finite elements in the dynamic analysis involving non - reflecting boundaries are introduced. A non - dimensional analysis on the ratio of wave length to the element size is carried out to clarify the amount of error involved in each solution. The application of different mass matrices, lumped, average and consistent, is examined in relation to the element size.

ÖZET

Dinamik hesaplarda 'yansıtmıyan sınır'ların kullanılması halinde eleman boyutları üzerindeki sınırlamalar ortaya konulmaktadır. Çözümlerdeki hata miktarlarını açıklamak için dalga boyunun eleman boyuna oranı üzerinde boyutsuz analizler yapılmıştır. Kullanılan eleman boyutuna bağlı olarak, tekil, orlama, ve yayılı kütle matrislerinin tatbiki incelenmektedir.

1. INTRODUCTION

The application of the finite element method to a wave propagation problem requires to choose size of elements conveniently. To decrease

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element sizes would increase the accuracy, but, this would mean, in general, more degrees of freedom leading costly and undesirable solution. In addition, the accuracy of this method for the solution of such a problem also depends upon the extent of the zone discretized. The larger the zone is, the better representation of the problem will be attained. To enlarge the finite element applied region with the intention of better approximation to the mathematical modelling, once again may create expensive computation.

However, Lysmer and Kuhlemeyer [1] have shown that an infinite media can be represented by a finite dynamical model. They used viscous boundaries in which normal and shear stresses are expressed as function of velocities at that particular point. Transmitting boundaries [2, 3] have also been used in the field of soil dynamics. In particular, these boundaries have been employed extensively in the dynamic soil - structure interaction analysis to simulate the effect of infinite left and right layered region on the finite element applied zone which is assumed to have a rigid boundaries at the base. Even though, the presence of transmitting boundaries on the both side of a mathematical model, again the computational procedure can be expensive in relation to depth of the soil deposit.

The non - reflecting, force, boundaries may be employed to overcome this difficulty. The properties of such boundaries may be summarized as follows :

1) These boundaries will transmit wave motion coming to the base,

2) Displacements at these boundaries will be the same to that of the same depth of the free field,

3) The total energy arrived to this boundary by a wave motion will be absorbed by viscous dampers.

4) Forces acting on the mathematical model along these boundaries will be due to the only incident wave component of the vertically travelling waves.

In the light of above discussion, the intention in this paper is, therefore, to chose element sizes as big as possible without deteriarating the accuracy of the solution when non - reflecting boundaries were utilised.

2. MATHEMATICAL MODEL FOR ERROR ANALYSIS

A continuous system which is discretized by finite elements can act as filter such that does not allow ebove a cutoff frequency to propagate. The upper bound of this frequency is associated with the form of the element mesh, size of element and the wave type. Therefore, size of elements will, in fact, has influence on the accuracy of the solution. To be able to define the relation between the size of element and the parameters involved in the solution, such as frequency, shear wave velocity, when non - reflecting boundaries were employed, will be of prime importance as far as the determination of error in the solution is concerned.

In spite of the cost the best way to determine any error involved in each solution, would be perhaps to reduce the elements sizes gradually. Some known values will be redetermined in each step for comparison. This comparison will help to express the amount of error as a function of the element size. Each time, to repeat this process is a very costly procedure to follow. Instead, a simply defined mathematical model can be utilised to attain the relationship between important parameters for a minimal error. A column study is, therefore suggested for this purpose. In such model, there is only, S waves or P waves. Displacements due to both waves are independent from each other and can be calculated separately. Also, the direction of wave propagation is known.

A soil solumn with infinite length is shown in Fig. 1.a. From this, a column with finite length H is considered, Fig. 1.b. At the bottom of this column, there, established the non - reflecting boundaries of which the properties has been described as above.

3. THE CALCULATION OF ERROR

The calculation of error can be performed in the following steps.

a) A harmonic displacement with a unit amplitude will be applied at the top free surface of the soil column. Then, the forces at the non - reflecting boundaries due to only incident wave components will be calculated by the use of the one dimensional wave equation.

b) The dynamic response at the free surface of this soil column will be recalculated under the effect of non - reflecting boundary forces and viscous damping forces with a defined frequency step.

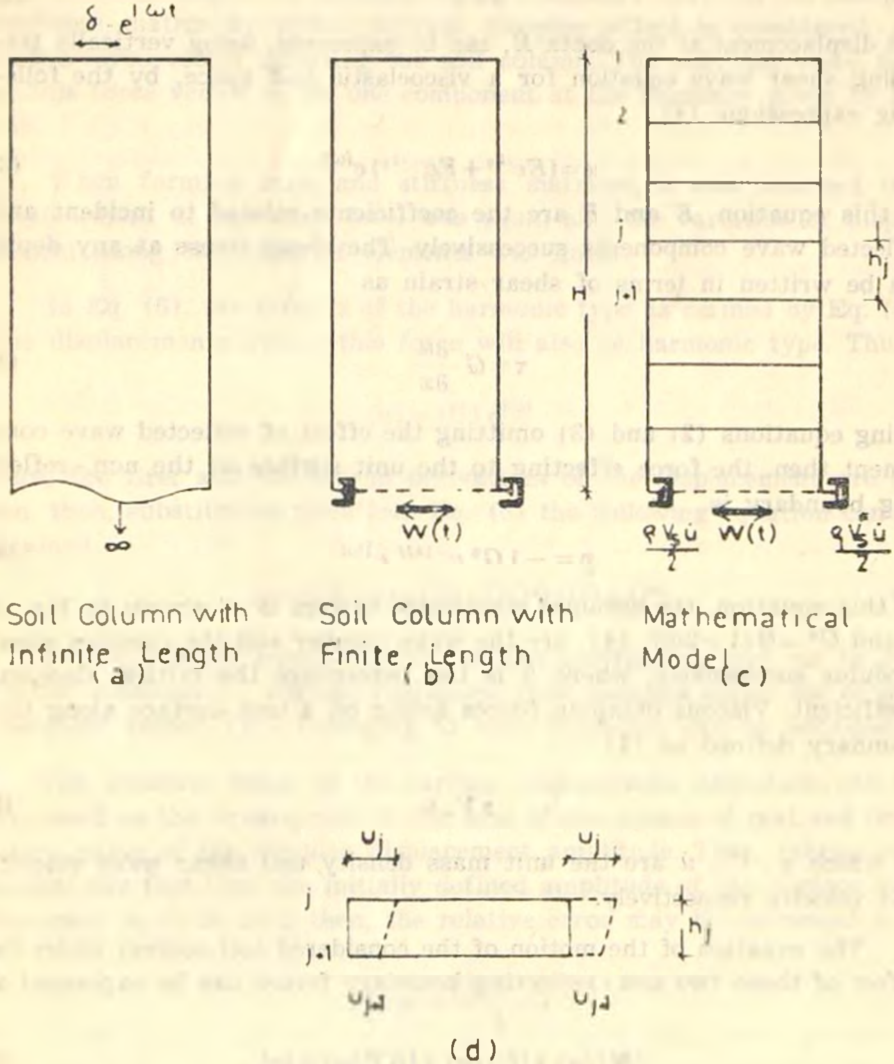


Fig. 1 Mathematical models for the error analysis.

c) The initial displacement with a unit amplitude will be compared with the calculated amplitude of the displacement on the free surface and relative error will be determined.

In the first step, a harmonic displacement wit unit amplitude in lateral direction can be expressed as

$$\delta = e^{i\omega t} \tag{1}$$

The displacement at the depth H, can be expressed, using vertically travelling shear wave equation for a viscoelastic half space, by the following expressions [4].

$$u = (Ee^{iky} + Fe^{-iky})\, e^{i\omega t} \tag{2}$$

In this equation, E and F are the coefficients related to incident and reflected wave components successively. The shear stress at any depth can be written in terms of shear strain as

$$\tau = G \frac{\partial u}{\partial x} \tag{3}$$

Using equations (2) and (3) omitting the effect of reflected wave component then, the force effecting to the unit surface at the non - reflecting boundary is

$$p = -1\, G^* e^{-1kH}\, e^{1\omega t} \tag{4}$$

In this equation, the assumed coordinate system is as shown in Fig. 1. k and $G^* \simeq G(1+2i\beta)$, [4], are the wave number and the complex shear modulus successively, where β is the percentage the critical damping coefficient. Viscous damping forces acting on a unit surface along this boundary defined as [1]

$$\rho V_s \dot{u} \tag{5}$$

in which ρ, V_s, $\dot{u}$ are the unit mass density and shear wave velocity and velocity respectively.

The equation of the motion of the considered soil column under the effect of these two non - reflecting boundary forces can be expressed as

$$[M]\{u\} + [C]\{\dot{u}\} + [K]^*\{u\} = \{p\} \tag{6}$$

In this equation $[M]$ is the mass matrix formed using both consistent and lumped mass matrices. Such as,

$$[M] = \alpha[M]_c + (1-\alpha) M_1 \tag{7}$$

in which $[M]_c$, $[M]_1$ and α are the consistent, lumped mass matrices and a scalar varying between $0<\alpha<1.0$ $[C]$ is the diagonal damping matrix. The only non - zero term in this matrix in ρV_s^* which corresponds to the degree of freedom at the boundary. $[K]^*$ is the complex stiffness matrix by which internal damping effect is considered. $\{p\}$ is the force vector effecting the soil column. The only non - zero term of this force vector is the one component at the boundary given by Eq. (4).

When forming mass and stiffness matrices, it was assumed that displacement at the same level was equal and the variation of displacement along the edges of elements was linceer.

In Eq. (6), the force is of the harmonic type as defined by Eq. (4). The displacements due to this force will also be harmonic type. Thus,

$$\{u\} = \{U\}\, e^{i\omega t} \tag{8}$$

Once, the first and the second derivatives of the displacement are taken, then, substituting them into Eq. (6) the following equation can be obtained.

$$(-\omega^2 [M] + i\omega [C] + [K]^*)\{U\} = \{P\} \tag{9}$$

This is a lincar equation with complex coefficients, and can be solved for a chosen ω_s, angular frequency, then, complex amplitude of displacement vector, $\{U\}_s$ belonging to each frequency can be obtained.

The absolute value of the surface displacement amplitude can be expressed as the square root of the sum of the square of real and imaginary parts of the complex displacement amplitude. Then, taking into account the fact that the initially defined amplitude of the surface displacement is to be unit, then, the relative error may be expressed as

$$error = \frac{|U| - 1}{1}$$

4. ERROR ANALYSIS

Non - dimensional error analyses were carried out among shear wave velocity, V_s. the height of finite elements, h, in the direction of wave propagation frequency, f, for the solution. In this analysis, the proportion of the wave length, $\lambda = V_s\, f$, to the height of an element, h, is defined as r, which is a non - dimensional parameter, is taken to be x - axis, and error is taken to be the y - axis.

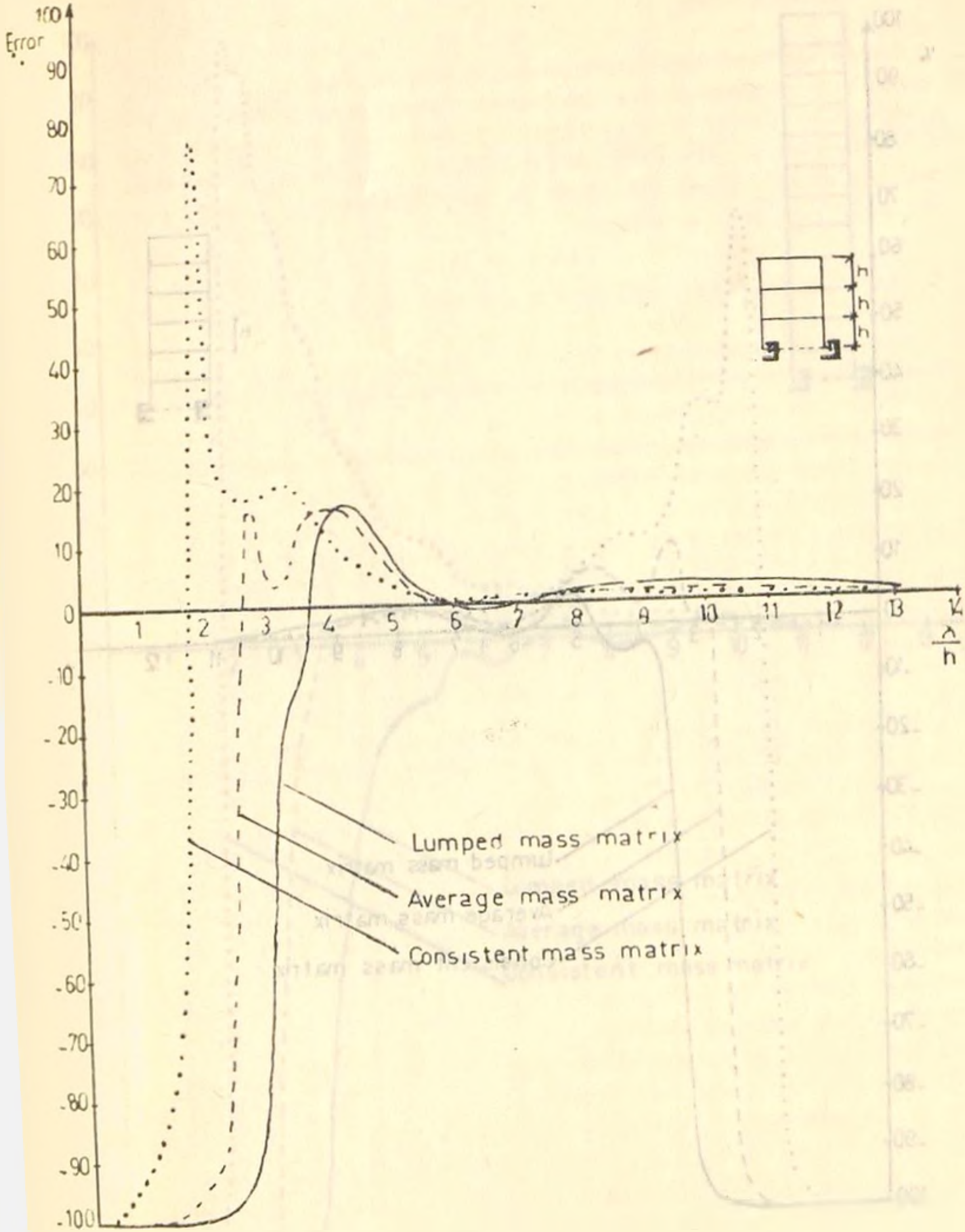
For numerical examples, a homogen soil column were considered of which the shear wave velocity, V_s, unit weight, Poisson's ratio and the percentage of the critical damping coefficient is taken 257 m/sn 2 ton/m^3, 0.35, 0.50, successively. Calculations were carried out on three different soil columns with the height of 30, 60 and 120 m. Each column was divided into 3, 6 and 12 rectangular elements successively with the equal height of 10 m. The analysis was performed for three type of mass matrices. These are namely, lumped, ($\alpha=0$), average ($\alpha=0.5$) and consistent ($\alpha=1$) mass matrices as defined in Eq. (7). Eq. (9), was solved for frequencies 1 to 10 Hz with the frequency step is being 0.1 and for the frequencies 10 - 100 Hz with the frequency step is being 0.5 Hz.

The results of the error analysis are presented in Fig. 2, 3 and 4. It has been seen that, the amount of error in each calculation becomes smaller as the value of $r=\lambda/h$ increases. Particularly, for cases, when heights of soil columns are 60 m and 120 m, the amount of error in each calculation approaches zero when r is bigger than 11. For the soil column of 30 m of height, error values become very small for the r values which are greater than 14.

In general, however, the wave length to height ratio is equal to 5, the amount of error have the value of around 10 %. Similar suggestion has also pointed out by Kunlemeyer and Lysmer [4].

When, consistent mass matrix were employed, it appears from the analysis that the error values approaches zero if the element size is as big as 1/2 to 1/1.8 of the wave length. To be able to choose the element size as big as half of the wave length, obviously would be very useful in practice. This is due to fact that it will help to reduce the number of degrees of freedom considerably and leads eventually cheaper solution. However, it can also be noticed that one should be very cautious to increase element size that big, since, this size may cause very erraneous results even for the r values around 2 as shown in Fig. 4. Therefore, it is not advisable to increase element size as much as half of the wave length.

Through the analysis of the results shown in Fig. 2, 3 and 4, it becomes clear that, in general, the ordinates of the error curve is smaller when average mass matrix was employed.



Error
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40
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-10
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-80
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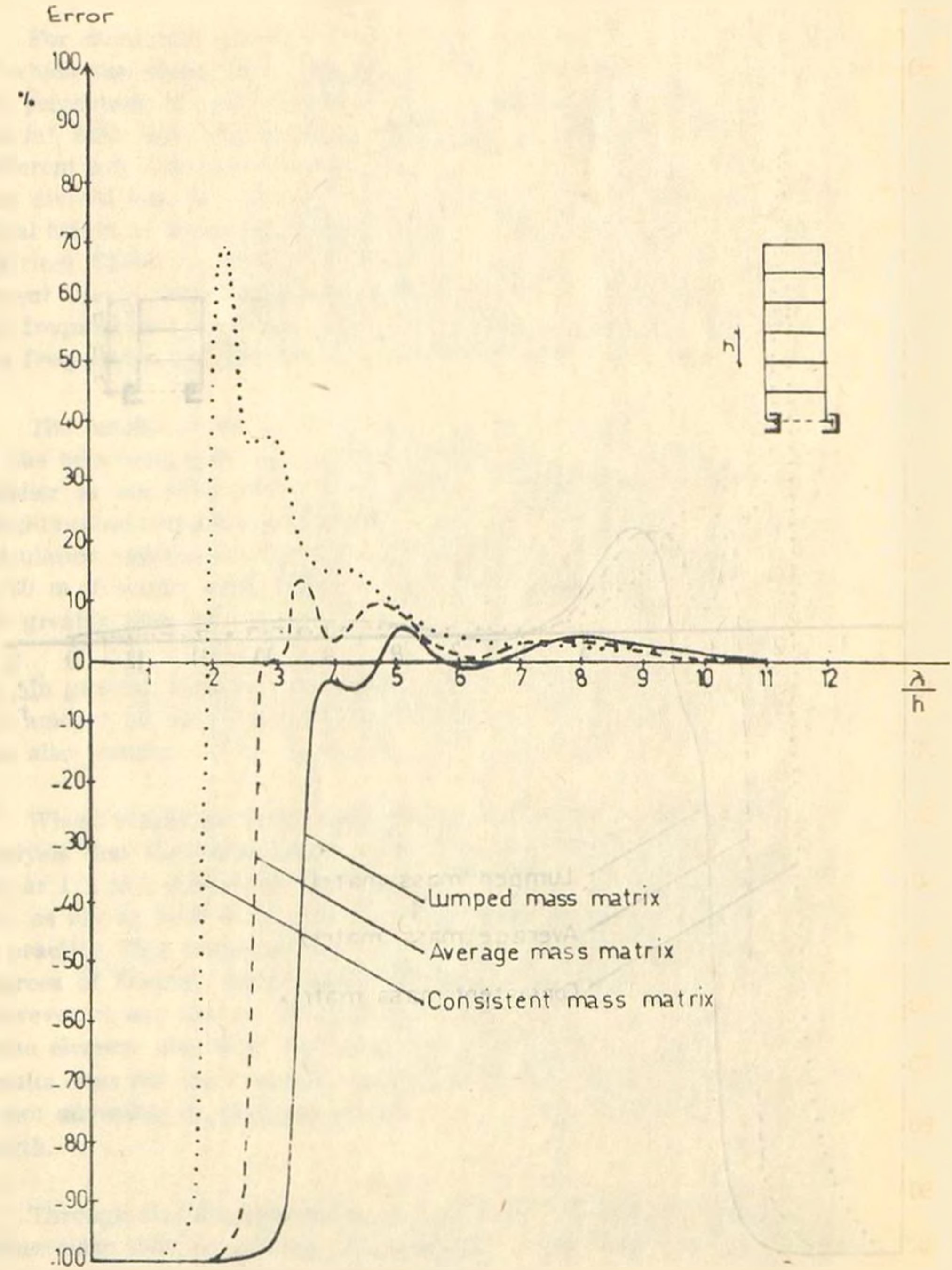
1 2 3 4 5 6 7 8 9 10 11 12 13 14
λ/h
h
h
h
Lumped mass matrix
Average mass matrix
Consistent mass matrix

Fig. 2


Error
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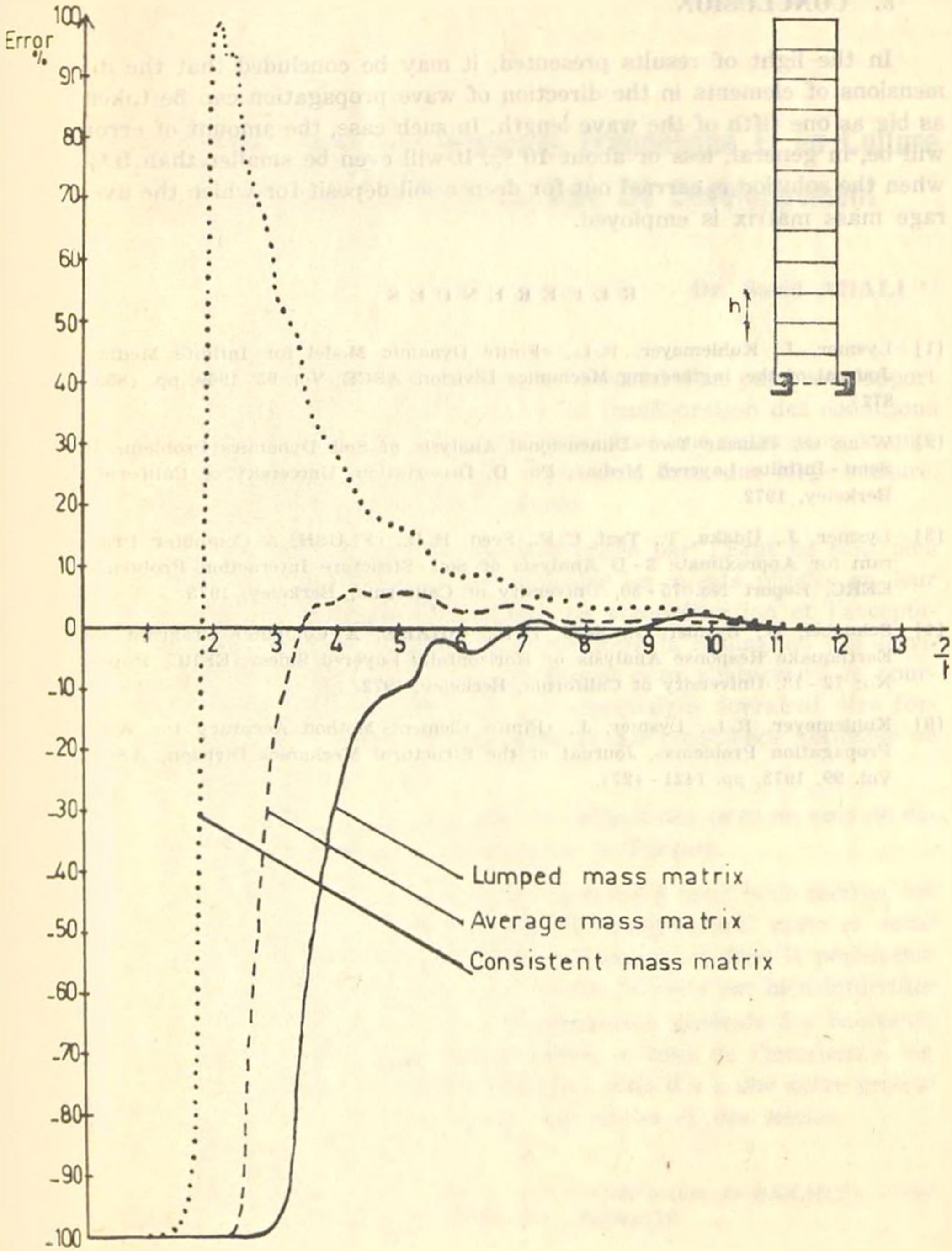
Lumped mass matrix
Average mass matrix
Consistent mass matrix

Fig. 3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λ/h
h
Lumped mass matrix
Average mass matrix
Consistent mass matrix

Fig. 4

5. CONCLUSION

In the light of results presented, it may be concluded that the dimensions of elements in the direction of wave propagation can be taken as big as one fifth of the wave length. In such case, the amount of error will be, in general, less or about 10 %. It will even be smaller than 5 % when the solution is carried out for deeper soil deposit for which the average mass matrix is employed.

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