Sabit kesitli bir boruda konsantrasyon dağılımının asimptotik hali

Asymptotic behaviour of the concentration distribution in a pipe of constant cross section

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Bu makalede, sabit kesitli bir boruda, konsantrasyon dağılımına ait momentler kullanılarak, konsantrasyon dağılımının asimptotik durumu incelenmiştir. Genel bir tetkik, ortalama konsantrasyonun, asimptotik halde, akımın ortalama hızıyla hareket eden bir noktaya göre normal dağılıma uyduğunu göstermiştir. Ayrıca, konsantrasyon momentlerinin sağladığı denklemlerin nasıl çözülebileceği açıklanmıştır.

In this paper, the asymptotic behaviour of the concentration distribution in a pipe of constant cross section, by use of moments of the concentration distribution, is considered. A general analysis shows that the mean concentration is ultimately distributed about a point which moves at the mean speed of the flow according to the normal distribution. Furthermore, the way to be followed, in the solutions of the equations satisfied by the moments of the concentration distribution, is explained.

1. Introduction

If a solute is injected into a solvent which is in a steady laminar flow through a circular pipe, it is dispersed longitudinally due to the variation in fluid over the cross section of the pipe interacting with lateral molecular diffusion and longitudinal molecular diffusion. Experimentally and theoretically it has been shown [1] that the combined effect of longitudinal convection and lateral diffusion is to disperse the solute longitudinally relative to a frame, which moves with the mean speed of the

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flow, by a process which is described by one - dimensional diffusion equation. This fact was refound by Aris [2] using the moments of the distribution of solute. The most important feature of the theory of dispersion given by Taylor [1] is that it enables one to describe the average concentration in a three - dimensional system by the solution of the one - dimensional diffusion equation. This fact has been confirmed by many authors both experimentally and theoretically (see references in [3]).

The analysis used for laminar flow has been extended to the cases of turbulent flow in a circular pipe [4] and turbulent flow in a wide channel with free surface [5]. A conclusion follows the fact that the combined action of turbulent lateral diffusion and convection by the mean flow, and longitudinal turbulent diffusion are ultimately to make the matter spread out symmetrically about a frame moving with the discharge velocity. A virtual diffusion coefficient may be defined if the statistical properties of the flow do not change within a cylindrical boundary (see discussion in [6]).

The present paper describes the application of the analysis used for laminar flow in a straight pipe [2] and for the flow between two parallel plates [7] to the case of turbulent flow in a pipe of constant but arbitrary cross section, taking into account a secondary flow over the cross section of the pipe. It is found that there are some similarities between the asymptotic behaviour of the higher moments of the concentration ditribution in a straight pipe of circular cross section and that in a pipe of constant but arbitrary cross section in which flow is three-dimensional.

The analysis used in the present paper shows that the mean concentration ultimately distributed about a point which moves at the mean speed of the flow according to the normal law of error, regardless the initial distribution of the concentration. Although the results given in the present paper are in the case of turbulent flow, they can be readily applied to the case of laminar flow.

2. Equation of turbulent diffusion

For an incompressible turbulent flow, far from the laminar region, the concentration is given by the equation

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0, \tag{2.1}$$

where c is the instantaneous value of concentration and $\mathbf{u}(\mathbf{x}, t)$ is the instantaneous value of velocity. Substituting c and $\mathbf{u}$, $c = C + c'$ and $\mathbf{u} = \mathbf{v} + \mathbf{v}'$ in equation (2.1), and then taking the average of it one finds

$$\frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = -\nabla \cdot (\overline{\mathbf{v}' c'}) \tag{2.2}$$

where C and v are the mean values of the concentration and the velocity respectively, and prime denotes the fluctuating quantities. In the case of the diffusion in a pipe of constant cross section, the turbulent diffusion flux can be written as [8].

$$-\nabla \cdot (\overline{\mathbf{v}' c'}) = \nabla_s \cdot (\varepsilon \nabla_s C) + \frac{\partial}{\partial x}\left(\varepsilon^* \frac{\partial C}{\partial x}\right),$$

where ε an ε^*, which depend only on the cross sectional variables, are diffusivities ; and s denotes the cross sectional derivatives and x is the coordinate which is taken along the pipe. Thus, equation (2.2) has the following from

$$\frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = \nabla_s \cdot (\varepsilon \nabla_s C) + \frac{\partial}{\partial x}\left(\varepsilon^* \frac{\partial C}{\partial x}\right),$$

or

$$\frac{\partial C}{\partial t} + \mathbf{v}_s \cdot \nabla_s C + u^* \frac{\partial C}{\partial x} = \nabla_s \cdot (\varepsilon \nabla_s C) + \left(\varepsilon^* \frac{\partial C}{\partial x}\right), \tag{2.3}$$

where $\mathbf{v}_s$ is the cross sectional velocity and u^* is the axial velocity which depend only on the cross sectional variables ; the general case will not be considered in the present paper.

Since the velocity components and the diffusivities depend only on the cross sectional variables, using the continuity equation for velocity field, equation (2.3) can be written as

$$\frac{\partial C}{\partial t} + \nabla_s \cdot (C \mathbf{v}_s) + \frac{\partial}{\partial x}(C u^*) = \nabla_s \cdot (\varepsilon \nabla_s C) + \frac{\partial}{\partial x}\left(\varepsilon^* \frac{\partial C}{\partial x}\right). \tag{2.4}$$

It is convenient to write equation (2.4) in a frame which moves at the mean speed of the flow. For this we put

$$X = x - U_m t \quad , \quad \tau = t \ .$$

where $X = X\ (x, t)$, $\tau = \tau\ (x, t)$ and U_m is the mean velocity. Using the properties of partial derivatives we have

$$\frac{\partial C}{\partial x} = \frac{\partial C}{\partial X}, \qquad \frac{\partial C}{\partial t} = \frac{\partial C}{\partial \tau} - u_m \frac{\partial C}{\partial X};$$

and inserting them into equation (2.4) we find

$$\frac{\partial C}{\partial t} + \nabla_s \cdot (C\mathbf{v}_s) + \frac{\partial}{\partial X}(uC) = \nabla_s \cdot (\varepsilon \nabla_s C) + \frac{\partial^2}{\partial X^2}(\varepsilon^* C), \tag{2.5}$$

where $u = u^* - U_m$ is the velocity with respect to the moving frame and it has zero mean, and τ is replaced by t.

The boundary and the initial conditions are

$$\varepsilon \frac{\partial C}{\partial n} = \text{ at wall and } C(A,X,0) = C_0(A,X), \tag{2.6}$$

respectively ; where A represents the cross sectional variables and $\partial/\partial n$ denotes the normal derivative to wall.

3. The moments of the concentration distribution

The q *th* moment of the concentration distribution is given by

$$C^{(q)}(A,t) = \int_{-\infty}^{\infty} X^q C(A,X,t)dX. \tag{3.1}$$

The zero order moment is related to the total mass of the matter, the first order moment is related to the position of the centre of mass, the second order moment is related to the variance of the distribution of solute, the third order moment is related to the skewness of the distribution and the fourth order moment is related to the kurtosis of the distribution. In order to have more knowledge about the distribution of solute the higher more than the fourth will be necessary [2].

In order to obtain the equations satisfied by the momenets of the distribution of solute let us multiply equation (2.5) by X^q and integrate with respect to X in the interval $(-\infty, +\infty)$, by the use of the boundary conditions for the concentration we obtain (see Appendix A)

$$\frac{\partial C^{(q)}}{\partial t} + \nabla_s \cdot (C^{(q)}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(q)}) = quC^{(q-1)} + q(q-1)\varepsilon^* C^{(q-2)}; \tag{3.2}$$

where it is assumed that the sufficient conditions on the behaviour of the concentration at both ends of the cloud of the solute are satisfied. The boundary conditions given by (2.6) can be written in terms of the moments as

$$\varepsilon \frac{\partial C^{(q)}}{\partial n} = 0 \qquad \text{at wall,} \quad C^{(q)}(A,0) = C_0^{(q)}(A). \tag{3.3}$$

It is possible to estimate the variations of the moments using the mean values of the moments of the concentration. The average of the q *th* moment of concentration is given by

$$C_m^{(q)} = \frac{1}{S} \int_S C^{(q)}(A\ t)dS, \tag{3.4}$$

where S denotes the cross section of the pipe, and it is assumed constant.

Let us take the average of all quantities in equation (3.2) over the cross section and the use of the integral identities (see Appendix B) we obtain

$$\frac{dC_m^{(q)}}{dt} = q\{uC^{(q-1)}\}_m + q(q-1)\{\varepsilon^{0}C^{(q-2)}\}_m \ ; \tag{3.5}$$

where $\{\quad\}_m$ shows the mean of any quantity.

Aris [2] showed for a straight pipe that the first two moments are ultimately sufficient to describe the concentration distribution. However, in order to get more information such as skewness and kurtosis, it is necessary to find the third and the fourth and also the higher moments of concentration.

4. The solutions of the moment equations

In this paragraph, first we write the equation (3.2) for q = 0 , 1 , 2 , ... , n , ... and then we find the dependence of the concentration moments, up to n *th* order, on t. Furthermore, here, we show that it is possible to separate the concentration moments two parts in which one part depends only on the cross sectional variables and the other depends on the cross sectional variables and time.

4.1. The zero order solution

For q = 0 equation (3.2) becomes

$$\frac{\partial C^{(0)}}{\partial t} + \nabla_s \cdot (C^{(0)}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(0)}) = 0, \tag{4.1}$$

and equation (3.5) behaves

$$\frac{dC_m^{(0)}}{dt} = 0 \ ; \tag{4.2}$$

the boundary conditions (3.3) take the forms

$$\varepsilon \frac{\partial C_m^{(0)}}{\partial n} = 0 \quad \text{at wall,} \quad C^{(0)}A.0) = C_0^{(0)}(A). \tag{4.3}$$

Eqation (4.2) gives $C^{(0)}_m$=constant. This means that the total mass of solute is conserved in all times. We take this constant as unity without any loss of generality. Thus $C^{(0)}$ ultimately goes to unity and it can be written as

$$C^{(0)} = g_{00} + O(A, t), \tag{4.4}$$

where g_{00} is a constant and from the definition of the average we find that $g_{00}=1$; $O(A, t)$ denotes a function which depends on the cross sectional variables and time, and when t goes to infinity this function goes to zero. For the purpose here we do not need the explicit form of it. As it has been explained previously we see from equation (4.4) that, the part which depends only on the cross sectional variables and the part which depends on time are superposable.

4.2. The first order solution

For q=1 equation (3.2) becomes

$$\frac{\partial C^{(1)}}{\partial t} + \nabla_s \cdot (C^{(1)}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(1)}) = uC^{(0)}, \tag{4.5}$$

and equation (3.5) behaves

$$\frac{dC_m^{(1)}}{dt} = \{uC^{(0)}\}_m \; ; \tag{4.6}$$

the boundary conditions (3.3) take the forms

$$\varepsilon \frac{\partial C^{(1)}}{\partial n} = 0 \quad \text{at wall,} \quad C^{(1)}(A,0) = C_0^{(1)}(A). \tag{4.7}$$

Since $C^{(0)}$ ultimately goes to unity and $\{u\}_m=0$, then we obtain asymptotically $C^{(1)}_m$=constant. Thus, we have

$$C^{(1)} = f_{00}(A) + O(A, t), \tag{4.8}$$

where $f_{00}(A) = f(A)$ satisfies the equation

$$\nabla_s \cdot (f\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s f) = u, \tag{4.9}$$

and the boundary condition for f(A) becomes

$$\varepsilon \frac{\partial f}{\partial n} = 0 \text{ at wall.}$$

The solution of eqution (4.9) subjected to the boundary condition depends on the explicit forms of u , v_s and ε ; however, we do not discuss it here for the purpose in the present paper.

As it has been previously explained, the first order moment of the concentration distribution is related to the centre of the total mass of mater. Since equation (4.6) gives asymptotically $C^{(1)}{}_m$=constant, the position of the centre of mass does not change according to a point which moves at the mean speed of the flow. This means that the centre of mass ultimately moves at the mean speed of the flow.

4.3. The second order solution

For q=2 equation (3.2) becomes

$$\frac{\partial C^{(2)}}{\partial t} + \nabla_s \cdot (C^{(2)} v_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(2)}) = 2u\, C^{(1)} + 2\varepsilon^* C^{(0)} \tag{4.10}$$

and equation (3.5) behaves

$$\frac{dC_m^{(2)}}{dt} = 2\{u\, C^{(1)}\}_m + 2\{\varepsilon^* C^{(0)}\}_m ; \tag{4.11}$$

the boundary conditions (3.3) take the forms

$$\varepsilon \frac{\partial C^{(2)}}{\partial n} = 0 \text{ at wall, } C^{(2)}(A,0) = C_0^{(2)}(A). \tag{4.12}$$

From equation (4.4) and equation (4.8), equation (4.11) ultimately becomes

$$\frac{dC^{(2)}}{dt} = 2\{uf\}_m + 2\{\varepsilon^*\}_m .$$

The expression of $C^{(2)}{}_m$ suggests

$$C^{(2)} = tg_{11} + g_{10}(A) + O(A, t), \tag{4.13}$$

where g_{11}=2ϰ and ϰ is given by

$$\varkappa = \{uf\}_m + \{\varepsilon^*\}_m . \tag{4.14}$$

When t goes to infinity the dominant term in equation (4.13) becomes and this gives the variance of the distribution. In other words, the half of the derivative of the variance with respect to t equals to the apparent diffusion coefficient. Without loss of generality, $g_{10}(A)$ can be taken as a function which has a zero mean. In order to obtain the equation satisfied by g_{10}, substituting equation (4.13) into equation (4.10) and using $\nabla \cdot \mathbf{v}_s = 0$ one finds

$$\nabla_s \cdot (g_{10}\,\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s g_{10}) = 2uf + 2(\varepsilon^* - \varkappa), \tag{4.15}$$

where g_{10} sattisfies the condition

$$\varepsilon \frac{\partial g_{10}}{\partial n} = 0 \text{ at wall.}$$

4.4. The third order solution

For $q=3$ equation (3.2) becomes

$$\frac{\partial^{(3)}C}{\partial t} + \nabla_s \cdot (C^{(3)}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(3)}) = 3uC^{(2)} + 3!\,\varepsilon^* C^{(1)}, \tag{4.16}$$

and equation (3.5) behaves

$$\frac{dC_m^{(3)}}{dt} = 3\{uC^{(2)}\}_m + 3!\,\{\varepsilon^* C\}_m\,; \tag{4.17}$$

the boundary conditions (3.3) take the forms

$$\varepsilon \frac{\partial C^{(3)}}{\partial n} = 0 \quad \text{at wall,} \quad C^{(3)}(A,0) = C_0^{(3)}(A).$$

From equation (4.8) and equation (4.13), equation (4.17) ultimately becomes

$$\frac{dC_m^{(3)}}{dt} = 3\{ug_{10}\}_m + 3!\{\varepsilon^* f\}_m\,.$$

The expression of $C^{(3)}_m$ suggests

$$C^{(3)} = t\,f_{11}(A) + f_{10}(A) + O(A,t)\,. \tag{4.18}$$

f_{11} is eqal to the sum of $3ug_{10} + 3!\varepsilon^* f$ and a function which has a zero mean. Substituting $C^{(3)}$, $C^{(2)}$ and $C^{(1)}$ into equation (4.16) one finds Equali-

$$\nabla_s \cdot (f_{11}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s f_{11}) = 3!\,u\varkappa, \tag{4.19}$$

$$f_{11} + \nabla_s \cdot (f_{10}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s f_{10}) = 3ug_{10} + 3!\,\varepsilon^* f.$$

zation of the coefficient of t to zero gives the first equation and equalization of the terms independent of time to zero gives the second equation. We do not need the explicit form of f_{10} for the purpose in the present paper ; however, it can be chosen as a function which has zero mean.

We need the explicit form of f_{11}. Comparing equation (4.19) with equation (4.9) we get

$$f_{11} = 3!\varkappa f.$$

Substituting the form of f_{11} into equation (4.18) one finds

$$C^{(3)} = 3!\varkappa t f(A) + f_{10}(A) + O(A, t). \tag{4.20}$$

4.5. The fourth order solution

For q = 4 equation (3.2) becomes

$$\frac{\partial C^{(4)}}{\partial t} + \nabla_s \cdot (C^{(4)}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(4)}) = 4uC^{(3)} + 12\varepsilon^* C^{(2)}, \tag{4.21}$$

and equation (3.5) behaves

$$\frac{dC_m^{(4)}}{dt} = 4\{uC^{(3)}\}_m + 12\{\varepsilon^* C^{(2)}\}_m ; \tag{4.22}$$

the boundary conditions (3.3) take the forms

$$\varepsilon \frac{\partial C^{(4)}}{\partial n} = 0 \quad \text{at wall,} \quad C^{(4)}(A,0) = C_0^{(4)}(A).$$

From equation (4.20) and equation (4.13), equation (4.22) ultimately becomes

$$\frac{dC_m^{(4)}}{dt} = 3!\,4\varkappa\{uf\}_m t + 4\{uf_{10}\}_m + 12 \times 2\varkappa\{\varepsilon^*\}_m t + 12\{\varepsilon^* g_{10}\}_m$$

$$= 4!\,\varkappa[\{uf\}_m + \{\varepsilon^*\}_m]t + [4\{uf_{10}\}_m + 12\{\varepsilon^* g_{10}\}_m]$$

$$= 4!\,\varkappa^2 t + [4\{uf_{10}\}_m + 12\{\varepsilon^* g_{10}\}_m] \cdot$$

Thus, $C^{(4)}_m$ has quadratic form in terms of t. The expression of $C^{(4)}_m$ suggests

$$C^{(4)}=t^2 g_{22}+t g_{21}(A)+g_{20}(A)+O(A,t), \tag{4.23}$$

where $\{g_{22}\}_m = !\varkappa^2/2$. Substituting $C^{(4)}$, $C^{(3)}$ and $C^{(2)}$ into equation (4.21) one finds

$$\nabla_s \cdot (g_{22}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s g_{22}) = 0,$$

$$2g_{22} + \nabla_s \cdot (g_{21}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s g_{21}) = 4uf_{11} + 12\varepsilon^* g_{11},$$

$$g_{21} + \nabla_s \cdot (g_{20}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s g_{20}) = 4uf_{10} + 12\varepsilon^* g_{10},$$

due to the uniformity condition for t goes to infinity. The term with t^2 gives the first equation, the term with t does the second and the term independent of time does the third. From the first equation and the boundary condition we obtain that g_{22} is constant. Thus g_{22} equals to its mean, namely $g_{22}=4!\varkappa^2/2$. Substituting the values of g_{22}, f_{11} and g_{11} into the second equation one finds

$$\nabla_s \cdot (g_{21}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s g_{21}) = 4!\,\varkappa(uf+\varepsilon^*) - 4!\,\varkappa^2 \cdot$$

It is clear that if we take the average of this equation the left hand side becomes zero by use of the integral identities and the boundary conditions, and the right hand side equals to zero because of $\{uf+\varepsilon^*\}_m=\varkappa$. However, we do not need the explicit form of g_{21} for the purpose in the present paper.

4.6. *The fifth order solution*

For q=5 equation (3.2) becomes

$$\frac{\partial C^{(5)}}{\partial t} + \nabla_s \cdot (C^{(5)}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(5)}) = 5uC^{(4)} + 20\varepsilon^* C^{(3)}, \tag{4.24}$$

and equation (3.5) behaves

$$\frac{dC_m^{(5)}}{dt} = 5\{uC^{(4)}\}_m + 20\{\varepsilon^* C^{(3)}\}_m; \tag{4.25}$$

the boundary conditions (3.3) take the forms

$$\varepsilon \frac{\partial C^{(5)}}{\partial n} = 0 \quad \text{at wall,} \quad C^{(5)}(A,0) = C_0^{(5)}(A) \cdot$$

Substituting the quantities appearing on the right hand side of equation (4.25) one finds

$$\frac{dC_m^{(5)}}{dt} = \frac{5!}{2} x^2t^2\{u\}_m + 5t\{ug_{21}\}_m + 5\{ug_{20}\}_m + 5!\, xt\{\varepsilon^* f\}_m f + 20\{\varepsilon^* f_{10}\}_m$$

$$= [5\{ug_{21}\}_m + 5!\, x\{\varepsilon^* f\}_m]t + [5\{ug_{20}\}_m + 20\{\varepsilon^* f_{10}\}_m],$$

where $\{u\}_m = 0$. $C^{(5)}_m$ has a quadratic form in terms of t. From the form of $C^{(5)}_m$ we can estimate

$$C^{(5)} = t^2 f_{22}(A) + t f_{21}(A) + f_{20}(A) + O(A,t), \tag{4.26}$$

where $2f_{22}$ is equal to the sum of $5ug_{21} + 5!x\varepsilon^* f$ and a function with zero mean. Substituting $C^{(5)}$, $C^{(4)}$ and $C^{(3)}$ into equation (4.24) one obtains

$$\nabla_s \cdot (f_{22}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s f_{22}) = \frac{5!}{2} x^2 u,$$

$$2f_{22} + \nabla_s \cdot (f_{21}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s f_{21}) = 5u\, g_{21} + 5!x\varepsilon^* f,$$

$$f_{21} + \nabla \cdot (f_{20}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s f_{20}) = 5u\, g_{20} + 20\, \varepsilon^* f_{10},$$

due to uniformity condition at infinity. The term with t^2 gives the first equation, the term with t does the second and the term independent of time does the third. Comparing the first equation with equation (4.9) we have $f_{22} = 5!x^2 f/2$. We do not need the explicit forms of the other functions appearing in equation (4.26). Thus we may write

$$C^{(5)} = \frac{5!}{2} x^2 f t^2 + f_{21} t + f_{20} + O(A,t) \cdot$$

It is possible to generalize the expressions so far we have obtained.

4.7. *The solution of any order*

From the expressions of the solutions obtained, up to the fifth order, the n *th* order and n+1 *th* order solutions can be deduced. We summarize the solutions obtained as

$$C^{(0)} = 1 + O(A,t),$$

$$C^{(1)} = f(A) + O(A,t),$$

$$C^{(2)} = 2xt + g_{10}(A) + O(A,t),$$

$$C^{(3)} = \frac{(2\times 2+1)!(2\times t)^1}{1!2^1} f(A) + f_{10}(A) + O(A,t),$$

$$C^{(4)} = \frac{(2\times 2)!(2\times t)^2}{2!2^2} + t\, g_{21}(A) + g_{20}(A) + O(A,t),$$

$$C^{(5)} = \frac{(2\times 2+1)!(2\times t)^2}{2!2^2} f(A) + t\, f_{21}(A) + f_{20}(A) + O(A,t) \cdot$$

The expressions given above suggest that, for example, the sixth order solution is in the form (see Appendix C)

$$C^{(6)} = \frac{(2\times 3)!(2\times t)^3}{3!2^3} + t^2 g_{32}(A) + t g_{31}(A) + g_{30}(A) + O(A,t) \cdot$$

We can estimate the n *th* term as

$$C^{(2n)} = \frac{(2n)!(2\times t)^n}{n!2^n} + \dots\, t\, g_{n1}(A_n) + g_{n0}(A) + O(A,t) \cdot$$

Since we assume that expression of $C^{(2n)}$ is in the form written above. we have to deduce the expression of n + *th* order solution.

For q = n + 1 equation (3.2) becomes

$$\frac{\partial C^{(2n+1)}}{\partial t} + \nabla_s \cdot (C^{(2n+1)} \mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(2n+1)})$$
$$= (2n+1) u C^{(2n)} + 2n(2n+1)\varepsilon^* C^{(2n+1)}$$

and equation (3.5) behaves

$$\frac{dC_m^{(2n+1)}}{dt} = (2n+1)\{uC^{(2n)}\}_m + 2n(2n+1)\{\varepsilon^* C^{(2n+2)}\}_m ;$$

the boundary condition (3.3) take the forms

$$\varepsilon \frac{\partial C^{(5)}}{\partial n} = 0 \quad \text{at wall,} \quad C^{(5)}(A,0) = C_0^{(5)}(A) \cdot$$

Since the dominant term of $C^{(2n)}$ is in order of t^n and the coefficient of t^n is constant, as it was seen from the equation satisfied by $C_m^{(2n+1)}$, the dominant term of $C_m^{(2n+1)}$ becomes in order of t^n . This discussion suggests

$$C^{(2n+1)}=t^n f_{nn}(A)+ \ldots . + t f_{n1}(A)+f_{n0}(A)+O(A,t) \cdot$$

Substituting this expression into equation satisfied by $C^{(2n+1)}$ gives

$$n f_{nn} t^{n-1}+t^n[\nabla_s \cdot (f_{nn}\mathbf{v}_s)-\nabla_s \cdot (\varepsilon \nabla_s f_{nn})]+ \cdots = \frac{(2n+1)!(2\varkappa t)_n}{n!2^n} u + \cdots$$

Equalizing the coefficient of the terms in order of t^n one finds

$$\nabla_s \cdot (f_{nn}\mathbf{v}_s) - \nabla_s \cdot (\varepsilon \nabla_s f_{nn}) = \frac{(2n+1)!(2\varkappa)^n}{n!2^n} u \cdot$$

Comparing this equation with equation (4.9) we have

$$f_{nn} = \frac{(2n+1)!(2\varkappa)^n}{n!2^n} f \cdot$$

Thus we may write

$$C^{(2n+1)}=\frac{(2+1)!(2\varkappa t)^n}{n!2^n} f(A)+ \cdots + t f_{n1}(A)+f_{n0}(A)+O(A.t) \cdot$$

The coefficients of the dominant terms in the experessions of the moments of even order are constants and those of the moments of odd order are functions which depend on the cross section. In the expression of any order moment, a term which depends only on the cross sectional variables always exists and the other terms depend on time. In this sense a linear separation always may be made. The part dependent on time can be subdivided in two parts, as one is a polinomial in terms of time and the other is a term which goes to zero when time goes to infinity. Such a separation has been used by many author (see for example [9]) without proof.

5. Comparison with the normal distribution

If we use the definition of the absolute skewness of the distribution (see for example [10]) as

$$\beta_1 = \frac{[C_m^{(3)}]^2}{[C_m^{(2)}]^3} \cdot$$

and we substitute in β_1 the asymptotic form which are

$$C_m^{(3)} \sim 6\varkappa\{f\}_m t \ , \ C_m^{(2)} \sim 2\varkappa t \ ,$$

we optain that β_1 ultimately varies as $1/t$; thus, any distribution of the solute tends to become more symmetrical.

We use the fourth moment to measure the degree to which a given distribution is flattened at its centre. This measure is given by

$$\beta_2 = \frac{C_m^{(4)}}{[C_m^{(2)}]^2} .$$

The asymptotic forms of $C_m^{(4)}$ and $C_m^{(2)}$ are

$$C_m^{(4)} \backsim 12\,\varkappa^2 t^2, \qquad C_m^{(2)} \backsim 2\,\varkappa\, t .$$

Substituting the values of $C_m^{(4)}$ and $C_m^{(2)}$ in β_2 we ultimately have

$$\beta_2 = 3 ,$$

which is a standard for the normal distribution.

The higher order skewness and kurtosis are given in the following forms

$$\beta_{2n-1} = \frac{[C_m^{(2n+1)}]^2}{[C^{(2)}]^{2n+1}} .$$

$$\beta_{2n-2} = \frac{C_m^{(2n)}}{[C_m^{(2)}]^n} .$$

respectively. The first is a measure of the skewness of the distribution and the second is related to the central flattened. For $n=1$ we have β_1 and for $n=2$ we have β_2 .

In the asymptotic case we have

$$C_m^{(2n+1)} \backsim \frac{(2n+1)!\,\varkappa^n \{f\}_m}{n!} t^n , \quad C_m^{(2)} \backsim 2\,\varkappa\, t .$$

Thus β_{2n-1} varies asymptotically as $1/t$. Therefore, the distribution becomes more symmetrical.

In the asymptotic case we may write

$$C_m^{(2n)} \backsim \frac{(2n)!\,\varkappa^n}{n!} t^n , \quad C_m^{(2)} \backsim 2\,\varkappa\, t .$$

Thus β_{2n-2} ultimately has the form

$$\beta_{2n-2} = \frac{(2n)!}{n!\,2^n}$$

or

$$\frac{(2n)!}{2^n n!} = \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots n(n-1)(n-2)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}{2n(2n-2)(2n-4)\cdots 10\ 8\cdot 6\ 4\cdot 2};$$

and canceling the even terms in the nominator and the denominator we find

$$\beta_{2n-2} = (2n-1)(2n-3)\cdots 3\,.$$

This gives the product of all odd terms. These are the relations which exist between the moments of the normal distribution (see for example [10]); and in this sense, the mean concentration is ultimately distributed about a point which moves with the mean speed of the flow according to the normal law of error.

APPENDIX A

$$X^q \frac{\partial C}{\partial t} + X^q \nabla_s \cdot (C\mathbf{v}_s) + X^q \frac{\partial (uC)}{\partial X} = X^q \nabla_s \cdot (\varepsilon \nabla_s C) + \varepsilon^* X^q \frac{\partial^2 C}{\partial X^2},$$

$$\frac{\partial}{\partial t}(X^q C) + \Delta_s \cdot (X^q C \mathbf{v}_s) + X^q \frac{\partial (uC)}{\partial X} = \nabla_s \cdot [\varepsilon \nabla_s (X^q C)] + \varepsilon^* \frac{\partial^2 C}{\partial X^2},$$

$$\frac{\partial}{\partial t}\int_{\infty -}^{\infty} X^q C dX + \nabla_s \cdot \left(\mathbf{v}_s \int_{-\infty}^{\infty} X^q C dX\right) + \int_{-\infty}^{\infty} X^q \frac{\partial (C)}{\partial X} dX$$

$$= \nabla_s \cdot \left[\varepsilon \nabla_s \left(\int_{-\infty}^{\infty} X^q C dX\right)\right] + \varepsilon^* \int_{-\infty}^{\infty} X^q \frac{\partial^2 C}{\partial X^2} dX\,.$$

$$\frac{\partial C^{(q)}}{\partial t} + \nabla_s \cdot (\mathbf{v}_s C^{(q)}) + [X^q C u]_{-\infty}^{\infty} - qu \int_{-\infty}^{\infty} X^{q-1} C dX$$

$$= \nabla_s \cdot (\varepsilon \nabla_s C^{(q)}) + \varepsilon^* \left[X^q \frac{\partial C}{\partial X}\right]_{-\infty}^{\infty} - \varepsilon^* q \int_{-\infty}^{\infty} X^{q-1} \frac{\partial C}{\partial X} dX\,,$$

$$\frac{\partial C^{(q)}}{\partial t} + \nabla_s \cdot (\mathbf{v}_s C^{(q)} - quC^{(q)} = \nabla_s \cdot (\varepsilon \nabla_s C^{(q)}) - \varepsilon^* q [X^{q-1} C]_{-\infty}^{\infty}$$

$$+\varepsilon^* q(q-1) \int_{-\infty}^{\infty} X^{q-2} C dX ,$$

$$\lim_{X \to \pm\infty} X^q C \to 0 , \quad \lim_{X \to \pm\infty} X^q \frac{\partial C}{X \partial} \to 0 ;$$

$$\frac{\partial C^{(q)}}{\partial t} + \nabla_s \cdot (\mathbf{v}_s C^{(q)} - quC^{(q-1)} = \nabla_s \cdot (\varepsilon \nabla_s C^{(q)}) + \varepsilon^* q(q-1) C^{(q-2)} ,$$

$$\lim_{X \to \pm\infty} X^{q-1} C \to 0 .$$

APPENDIX B

$$\frac{\partial}{\partial t}\left(\frac{1}{S} \int_S C^{(q)} dS\right) + \frac{1}{S} \int_S \nabla_s \cdot (C^{(q)} \mathbf{v}_s) dS - \frac{1}{S} \int_S \nabla_s \cdot (\varepsilon \nabla_s C^{(q)}) dS$$

$$= \frac{q}{S} \int_S u C^{(q-1)} dS + \frac{q(q-1)}{S} \int_S \varepsilon^* C^{(q-2)} dS,$$

$$\frac{dC_m^{(q)}}{dt} = q \{u C^{(q-1)}\}_m + q(q-1) \{\varepsilon^* C^{(q-)}\}_m,$$

$$\int_S \nabla_s \cdot (C^{(q)} \mathbf{v}_s) dS = \int_\Gamma \mathbf{n} \cdot \mathbf{v}_s C^{(q)} dl = 0,$$

since $\mathbf{n} \cdot \mathbf{v}_s = 0$ on Γ and

$$\int_S \nabla_s \cdot (\varepsilon \nabla_s C^{(q)}) dS = \int \mathbf{n} \cdot (\varepsilon \nabla_s C^{(q)}) dl$$

$$= \int_\Gamma \varepsilon \frac{\partial C^{(q)}}{\partial n} dl = 0,$$

since $\varepsilon \partial C / \partial n = 0$ on Γ, where Γ denotes the boundary, $\mathbf{n}$ is the unit normal vector of Γ and $\partial / \partial n$ is the normal derivative to wall.

APPENDIX C

For $q = 6$ equation (3.2) becomes

$$\frac{\partial C^{(6)}}{\partial t} + \nabla_s \cdot (C^{(6)} v_s) - \nabla_s \cdot (\varepsilon \nabla_s C^{(6)}) = 6u\, C^{(5)} + 30\, \varepsilon^* C^{(4)},$$

and eqation (3.5) behaves

$$\frac{dC_m^{(6)}}{dt} = 6\{u\, C^{(5)}\}_m + 30\, \{\varepsilon^* C^{(4)}\}_m.$$

Substituting $C^{(5)}$ and $C^{(4)}$ into equation satisfied by $C_m^{(6)}$ one finds

$$\frac{dC_m^{(6)}}{dt} = \frac{6!\,x^2}{2} (\{uf\}_m + \{\varepsilon^*\}_m) t^2 + \{6u\, f_{21} + 30\, \varepsilon^* g_{21}\}_m\, t + \{6u\, f_{20} + 30\, \varepsilon^* g_{20}\}_m$$

$$= \frac{6!\,x^3}{2}\, t^2 + \{6u\, f_{21} + 30\, \varepsilon^* g_{21}\}_m\, t + \{6u\, f_{20} + 30\, \varepsilon^* g_{20}\}_m\, .$$

From the form of $C_m^{(6)}$ we may estimate

$$C^{(6)} = t^3 g_{33}(A) + t^2 g_{32}(A) + t\, g_{31}(A) + g_{30}(A) + O(A, t)\, .$$

Comparison with the expression of $C_m^{(6)}$ gives $\{g_{33}\}_m = 6!x^3/6$. Substituting $C^{(5)}$ and $C^{(4)}$ into equation satisfied by $C^{(6)}$ one obtains

$$\nabla_s \cdot (g_{33}\, v_s) - \nabla_s \cdot (\varepsilon\, \nabla_s\, g_{33}) = 0,$$

$$3\, g_{33} + \nabla_s \cdot (g_{32}\, v_s) - \nabla_s \cdot (\varepsilon\, \Delta_s\, g_{32}) = 6u\, f_{22} + 30\, \varepsilon^*\, g_{22},$$

$$2\, g_{32} + \nabla_s \cdot (g_{31}\, v_s) - \nabla_s \cdot (\varepsilon\, \nabla_s\, g_{31}) = 6u\, f_{21} + 30\, \varepsilon^*\, g_{21},$$

$$g_{31} + \nabla_s \cdot (g_{30}\, v_s) - \nabla_s \cdot (\varepsilon\, \nabla_s \cdot (\varepsilon\, \nabla_s\, g_{30}) = 6u\, f_{20} + 30\, \varepsilon^*\, g_{20}\, .$$

due to uniformity condition at infinity. From the term with t^3 one obtains the first equation, from the term with t^2 one finds the second, from the term with t one has the third and from the term independent of time one obtains the fourth. The solution of the first equation subjected to the boundary condition is g_{33}=constant. Thus g_{33} equals to its mean, namely $g_{33} = 6!x^3/6$.

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