On the Lateral Buckling and Vibration of Elastic Beam Subjected to Conservative and Follower Loads

Konservatif ve İzleyici Yük Altındaki Elastik Kirişin Yanal Burkulması ve Titreşimi

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Lateral buckling and vibration of elastic beam with narrow rectangular strip under the combined action of concentrated, conservative and follower loads are investigated for two cases of boundary conditions. The convergence of the Galerkin's method is studied and the corresponding eigencurves are obtained at which Galerkin's method gives different approximations. The divergence and flutter loads of the problem are calculated and represented for various values of follower load in relation to the applied load.

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İnce dikdörtgen kesitli elastik bir kirişin düşey ve izleyici, tekil yük altındaki yanal burkulması ve titreşimi incelenerek, burkulma yükleri hesaplanmıştır.

Introduction

Lateral buckling of an elastic beam supported at the ends under vertical load at the middle section has been investigated in detail (1, 7, 10). In the these studies, it is assumed that the applied load remains vertical regardless of rotation of the cross section. Therefore, the load as well as the stability problem are conservative. The loss of the stability occurs at the statical position of the beam when the load reaches

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the buckling value, where the beam has a disturbed equilibrium position close to the undisturbed one, and the beam buckles by divergence. If this statical problem is investigated from a dynamical point of view — in this case the small vibration of the beam has to be taken into account —, the relationship between the load and the vibration frequency, which characterizes the eigencurves of the beam, is obtained. There, it can be seen more clearly that the static instability, the instability under conservative load, occurs at the point where the eigencurves intersect the load-axis. The eigencurves meet the frequency-axis at the points which correspond to the free vibration of the beam.

But, if load follows the rotation of the section, it is a follower load and the problem will be a nonconservative one. Such nonconservative problems can be studied by taking the vibration of the beam into account (2). If its eigencurve has the same form as that of conservative load, i.e., if it intersects the load-axis without having a maximum value for vibration frequency, the beam will buckle again by divergence although the load is nonconservative. The correspondding buckling load can be found also without considering the vibration of the beam. But, if the eigencurve does not intersect the load-axis, which means there is no value of the load for which there can exist a disturbed form of static equilibrium close to the undisturbed form, then the beam may buckle by flutter. Buckling by flutter will occur at the critical value of the frequency at which the two values of the frequency corresponding to a load coincide. With further increase in the load the mentioned values of the frequency become complex, and the flutter occurs because one of these has a negative imaginary part (4). For the approximate solution of the nonconservative problem the Galerkin's method, the convergence of which has been proved, can be applied (3, 5).

The stability of beam subjected to conservative or follower forces has been studied by many authors in detail (2, 7, 10). In order to see the difference between the conservative and nonconservative problems more detail, the two kinds of forces, the conservative and the follower ones, were considered to act together: the clamped-free column and simply supported rod subjected to its own weight and follower forces (using the Galerkin's method), beams with six typical cases of boundary conditions (using the finite difference method) (6, 8, 9).

The present study deals with the lateral buckling of an elastic beam subjected to conservative and also follower, concentrated forces with two boundary conditions. Besides this the convergence of the Galerkin's method is studied.

Statement of the problem

Consider a narrow rectangular strip of length l and height h supported at both ends and subjected to a concentrated, conservative force Q_c and follower force Q_f applied at the centroid of the middle cross section as shown in Fig. 1. The equation of motion of laterally deflected and twisted element of the beam are obtained from Fig. 2 as follows:

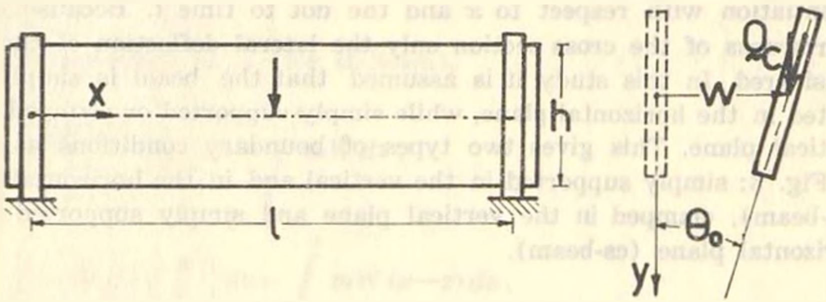

Fig. 1. Elastic beam under action of conservative and follower forces

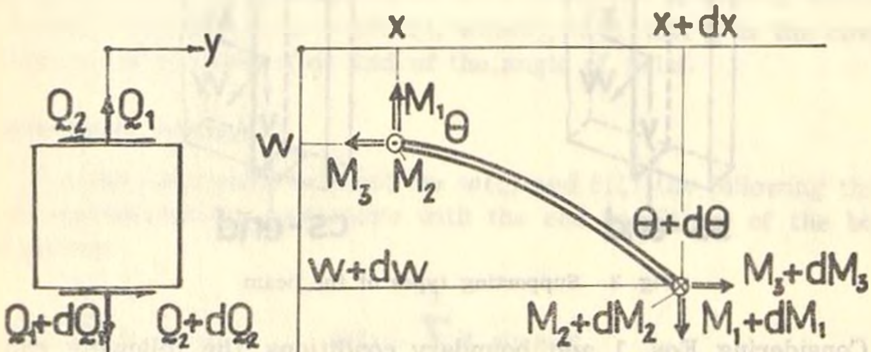

Fig. 2. Lateral deflected and twisted element of the beam

$$Q_1' - m\ddot{W} = 0,$$
$$Q_2' = 0,$$
$$M_3' - mr^2\ddot{\Theta} + Q_2 W' = 0,$$
$$M_1' - Q_2 = 0,$$
$$M_2' + Q_1 = 0, \qquad (1)$$

and relations of deformation are:

$$M_3 + M_1 W' = GJ\Theta',$$
$$M_2 - M_1\Theta = W'' EI/(1-\mu^2), \quad (2)$$

in which $W(x, t)$ = the lateral deflection and $\Theta(x,t)$ = the angle of twist of the cross section, m = the mass per unit length, $EI/(1-\mu^2)$ = the small bending stiffness, GJ = the torsional stiffness, r = the polar radius of inertia of the cross section, and the prime denotes here differentiation with respect to x and the dot to time t. Because of the narrowness of the cross section only the lateral deflection of the beam considered. In this study it is assumed that the beam is simply supported in the horizontal plane, while simply supported or clamped in the vertical plane. This gives two types of boundary conditions as shown in Fig. 3: simply supported in the vertical and in the horizontal planes (ss-beam), clamped in the vertical plane and simply supported in the horizontal plane (cs-beam).

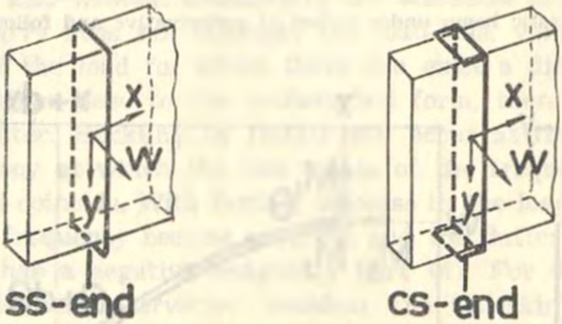

Fig. 3. Supporting types of the beam

Considering Eqs. 1 and boundary conditions, the following equations are obtained:

$$Q_1 = -0.5\alpha Q\Theta_0 - Q_m,$$
$$Q_2 = 0.5\,Q,$$
$$M_1 = -0.125\beta Ql + 0.5Qx,$$
$$M_2 = 0.5\alpha Q\Theta_0 x + M_m, \qquad \text{for } 0 \leqslant x \leqslant 0.5\,l$$

and

$$Q_1 = 0.5\alpha Q \Theta_0 - Q_m ,$$

$$Q_2 = -0.5\, Q ,$$

$$M_1 = 0.125\beta Q l + 0.5\, Q (l - x) ,$$

$$M_2 = 0.5\alpha Q \Theta_0 (l - x) + M_m , \qquad \text{for } 0.5\, l \leqslant x \leqslant l$$

where

$$Q = Q_c + Q_l ,$$

$$\alpha = Q_l / Q ,$$

$$\beta = 0 \text{ (for ss-beam)}, \; = 1 \text{ (for cs-beam)},$$

$$Q_m = \int_0^l m \ddot{W} \left(1 - \frac{z}{l}\right) dz - \int_0^x m \ddot{W}\, dz ,$$

$$M_m = x \int_0^l m \ddot{W} \left(1 - \frac{z}{l}\right) dz - \int_0^x m \ddot{W} (x - z)\, dz . \tag{3}$$

The equations of the problem can be freed from time by setting $\ddot{W}(x, t) \rightarrow - l\Omega^2 w(\zeta)$ and $\ddot{\Theta}(x, t) \rightarrow - \Omega^2 \theta(\zeta)$, where $\zeta = l/x$, and Ω is the circular frequency of the deflection and of the angle of twist.

Approximate solution

For the dimensionless functions $w(\zeta)$ and $\theta(\zeta)$ the following three-term approximations compatible with the end conditions of the beam, are chosen :

$$w(\zeta) = \sum_{n=1}^{3} w_n \sin n\pi\zeta , \qquad \theta(\zeta) = \sum_{n=1}^{3} \theta_n \sin n\pi\zeta . \tag{4}$$

With Eqs. 3 and 4 the function M_m given in Eq. 3 yields

$$M_m = - \frac{m\, \Omega^2 l^4}{\pi^2} \sum_{n=1}^{3} \frac{w_n}{n^2} \sin n\pi\zeta . \tag{5}$$

Substituting Eq. 1 in Eq. 2 gives

$$w'' - \frac{(1-\mu^2)l}{EI} M_2 + \frac{(1-\mu^2)l}{EI} \theta M_1 = 0 \ ,$$

$$\theta'' + \omega_t^2 \theta - \frac{l}{GJ} M_1 w'' = 0, \qquad (6)$$

where the prime denotes differentiation with respect to ζ. After substituting the function $w(\zeta)$ and $\theta(\zeta)$ given in Eq. 4 into Eq. 6 and applying the Galerkin's method, the following relations are obtained:

$$\pi^2 \int_0^1 \left(\sum_{n=1}^3 w_n n^2 \sin n\pi\zeta \right) \sin m\pi\zeta + 0.5(\theta_1 - \theta_3) q_b \alpha \left(\int_0^{0.5} \zeta \sin m\pi\zeta \cdot d\zeta + \right.$$

$$\left. + \int_{0.5}^1 (1-\zeta) \sin m\pi\zeta \cdot d\zeta \right) - \frac{\omega_b^2}{\pi^2} \int_0^1 \left(\sum_{n=1}^3 \frac{w_n}{n^2} \sin n\pi\zeta \right) \sin m\pi\zeta \cdot d\zeta +$$

$$+ 0.125 \beta q_b \int_0^1 \left(\sum_{n=1}^3 \theta_n \sin n\pi\zeta \right) \sin m\pi\zeta \cdot d\zeta - 0.5 q_b \int_0^{0.5} \left(\sum_{n=1}^3 \theta_n \sin n\pi\zeta \right)$$

$$\cdot \zeta \sin m\pi\zeta \cdot d\zeta - 0.5 q_b \int_{0.5}^1 \left(\sum_{n=1}^3 \theta_n \sin n\pi\zeta \right) (1-\zeta) \sin m\pi\zeta \cdot d\zeta = 0 \ ,$$

$$\pi^2 \int_0^1 \left(\sum_{n=1}^3 \theta_n n^2 \sin n\pi\zeta \right) \sin m\pi\zeta - \omega_t^2 \int_0^1 \left(\sum_{n=1}^3 \theta_n \sin n\pi\zeta \right) \sin m\pi\zeta \cdot d\zeta +$$

$$+ 0.125 \beta q_t \pi^2 \int_0^{0.5} \left(\sum_{n=1}^3 w_n n^2 \sin n\pi\zeta \right) \sin m\pi\zeta \ d\zeta -$$

$$- 0.5 q_t \pi^2 \int_0^{0.5} \left(\sum_{n=1}^3 w_n n^2 \sin n\pi\zeta \right) \zeta \sin m\pi\zeta \cdot d\zeta -$$

$$- 0.5 q_t \pi^2 \int_{0.5}^1 \left(\sum_{n=1}^3 w_n n^2 \sin n\pi\zeta \right) (1-\zeta) \sin m\pi\zeta \cdot d\zeta = 0 \ , \ \text{for} \ m = 1, 2, 3$$

where

$M =$

$\frac{1}{2\pi^2}(\pi^4-\omega_b{}^2)$	0	0	$\frac{q_b}{4}(\frac{1-4\alpha}{\pi^2}+\frac{1-\beta}{4})$	0	$-q_b\frac{1-4\alpha}{4\pi^2}$
0	$\frac{1}{8\pi^2}(16\pi^4-\omega_b{}^2)$	0	0	$q_b\frac{1-\beta}{16}$	0
0	0	$\frac{1}{18\pi^2}(81\pi^4-\omega_b{}^2)$	$-\frac{q_b}{36\pi^2}(9-4\alpha)$	0	$\frac{q_b}{4}(\frac{1-4\alpha}{9\pi^2}+\frac{1-\beta}{4})$
$\frac{q_t}{4}(1+\frac{\pi^2(1-\beta)}{4})$	0	$-\frac{9}{4}q_t$	$\frac{1}{2}(\pi^2-\omega_t{}^2)$	0	0
0	$q_t\pi^2\frac{1-\beta}{4}$	0	0	$\frac{1}{2}(4\pi^2-\omega_t{}^2)$	0
$-\frac{q_t}{4}$	0	$\frac{q_t}{4}(1+\frac{9\pi^2(1-\beta)}{4})$	0	0	$\frac{1}{2}(9\pi^2-\omega_t{}^2)$

$v = (w_1, w_2, w_3, \theta_1, \theta_2, \theta_3)$

Fig. 4. The matrix M and the vector v

$$\omega_b{}^2 = \frac{m\,\Omega^2 l^4 (1-\mu^2)}{EI}\,, \quad \omega_t{}^2 = \frac{m r^2 \Omega^2 l^2}{GJ}\,, \quad q_b = \frac{Q l^2 (1-\mu^2)}{EI}\,, \quad q_t = \frac{Q l^2}{GJ}\,.$$

These relationships yield a system of six linear homogen equations, which can be written in the matrix form as $M \cdot v = 0$. The matrix M and the vector v are given in Fig. 4. The determinant of M yields the stability condition, i.e., a relationship between the applied force and the frequency of the beam.

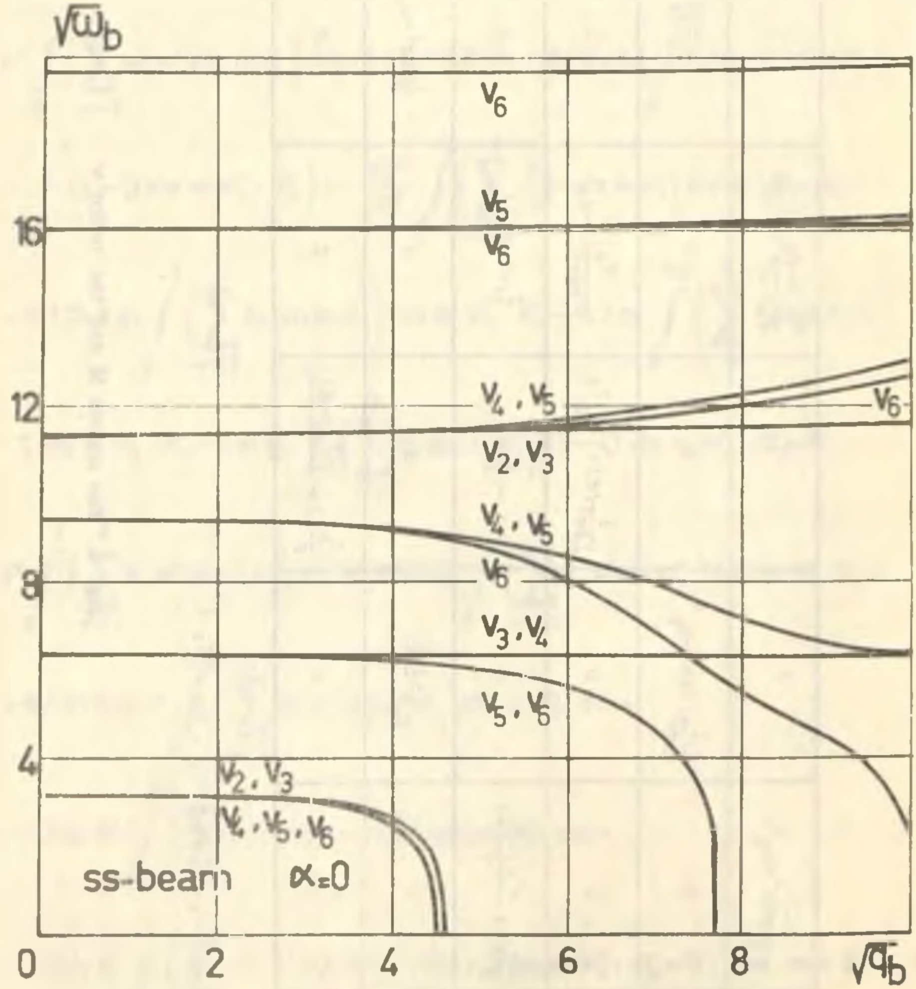

Fig. 5a. Eigencurves of the ss-beam for $\alpha=0$

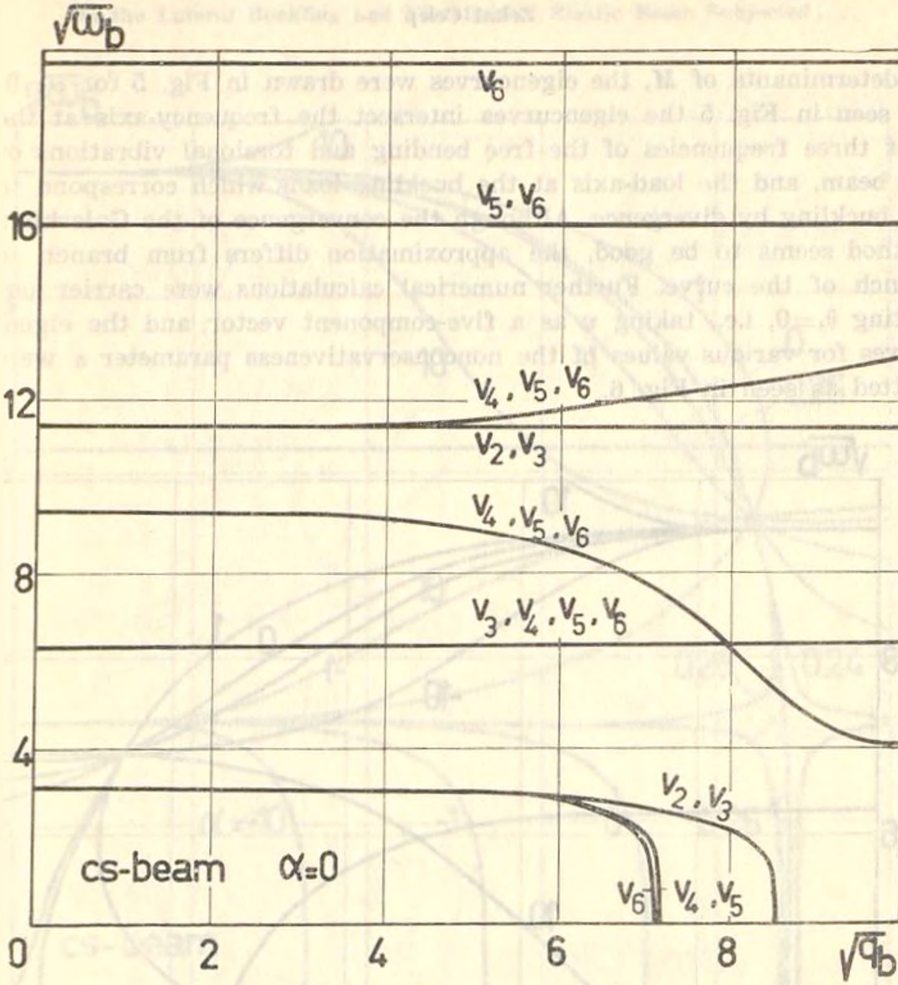

Fig. 5b. Eigencurves of the cs-beam for $\alpha=0$

The numerical procedure

In the numerical solution the following relations were used, remembering that the cross section is a narrow rectangular strip,

$$q_b=2(1-\mu)q_t\,, \qquad \omega^2_b=24(1-\mu)\omega^2_t/\lambda^2,$$

where $\lambda=h/l$. The numerical computation was made by setting $\mu=0.3$ and $\lambda=0.1$ on the B3700 computer at the Computing Center of the Technical University of Istanbul. The relation between q_b and ω_b was obtained and plotted to give the eigencurves of the beam. Assuming the vector v as having two, three, four, five and finally six components, i.e., $v_2=(w_1\,,\theta_1)$, $v_3=(w_1\,,w_2\,,\theta_1)$, $v_4=(w_1\;w_2\,,w_3\,,\theta_1)$, $v_5=(w_1\,,w_2\,,w_3\,,\theta_1\,,\theta_2)$ and $v_6=(w_1\,,w_2\,,w_3\,,\theta_1\,,\theta_2\,,\theta_3)$, and taking the corresponding subdeterminants of M, the eigencurves were drawn in Fig. 5 for $\alpha=0$. As seen in Fig. 5 the eigencurves intersect the frequency-axis at the first three frequencies of the free bending and torsional vibrations of the beam, and the load-axis at the buckling loads which correspond to the buckling by divergence. Although the convergence of the Galerkin's method seems to be good, the approximation differs from branch to branch of the curve. Further numerical calculations were carrier out setting $\theta_3=0$, i.e., taking v as a five-component vector, and the eigencurves for various values of the nonconservativeness parameter α were plotted as seen in Fig. 6.

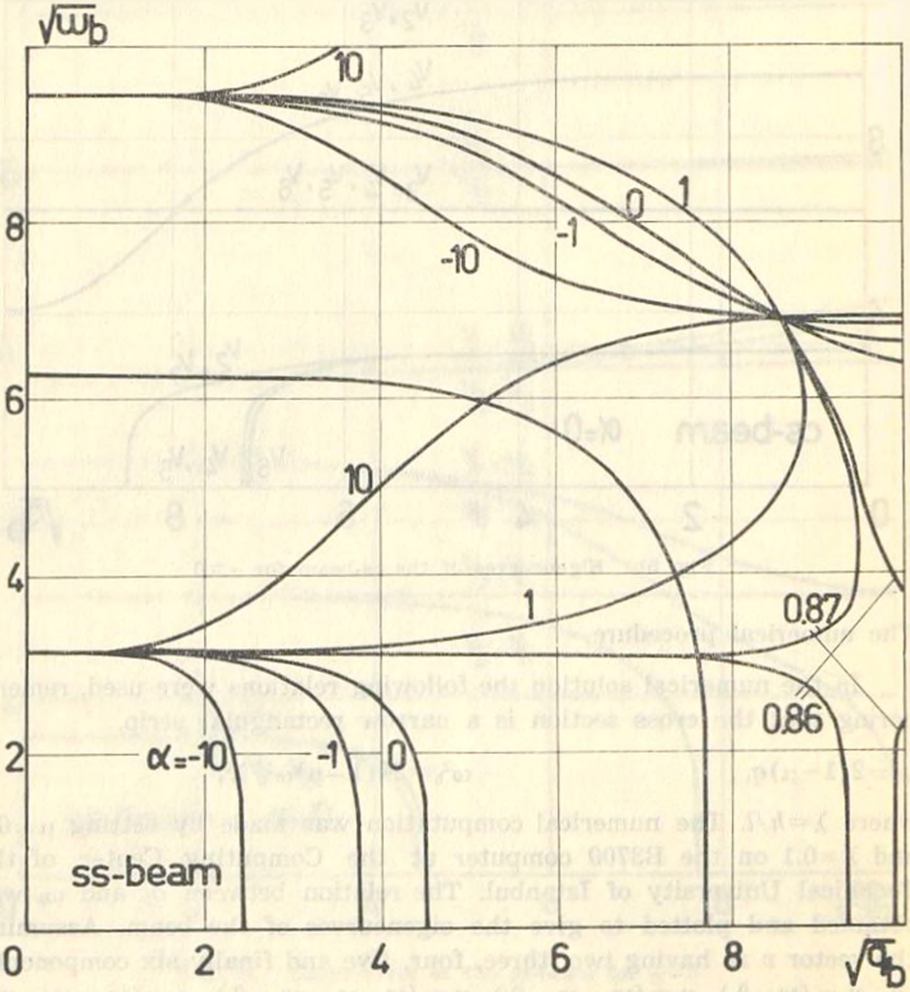

Fig. 6a. Eigencurves of the ss-beam for various values of α

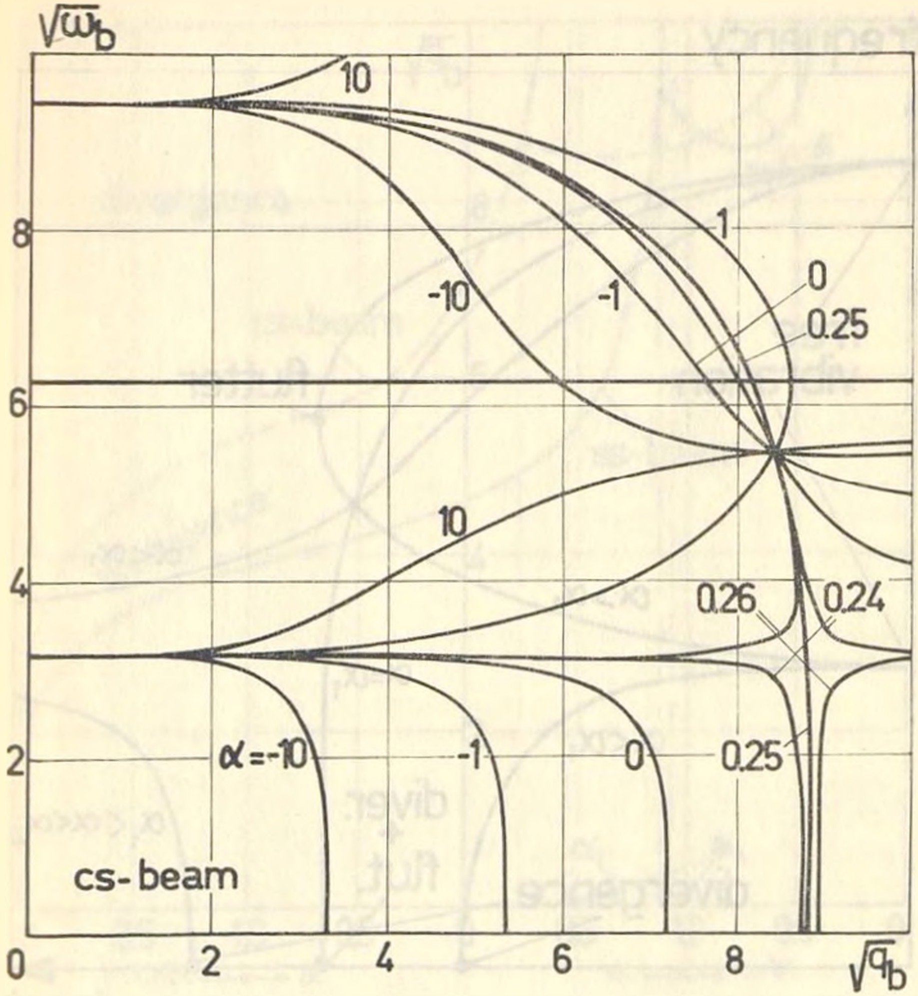

Fig. 6b. Eigencurves of the cs-beam for various values of α

Conclusions

The second eigencurves of the beams are independent of α, because the angle of twist of the middle cross section is zero at the second vibration mode. Thus, the ss-beam has a divergence load independent of α, while the cs-beam has not. The forms of the first and third eigencurves are represented in Fig. 7. If $\alpha<\alpha_1$, the beams have one divergence load only, while they have one divergence and one flutter load for $\alpha_1<\alpha<\alpha_2$. The divergence load vanishes for $\alpha>\alpha_2$. At $\alpha=\alpha_1$

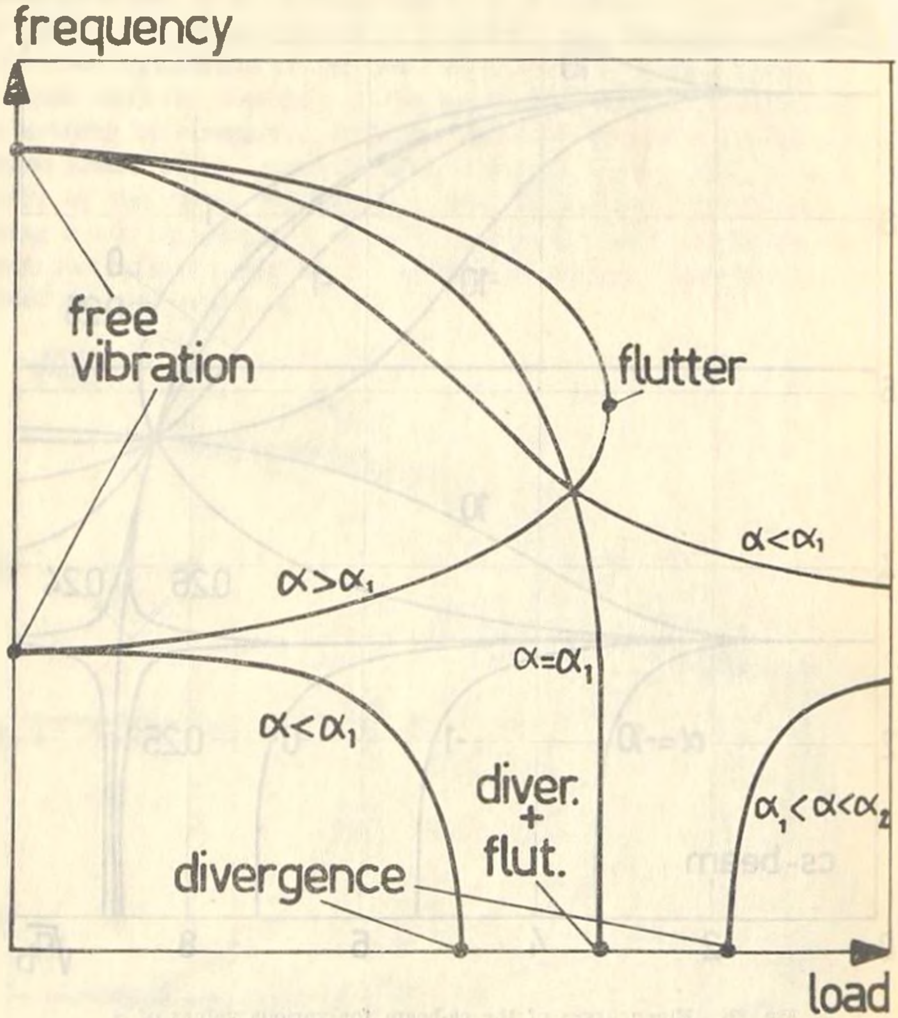

Fig. 7. Typical eigencurves of the beam

the two eigencurves coincide and take a vertical tangent at the point of intersection with the load-axis, and thus, the two critical loads become equal. The values of the lateral buckling loads versus the nonconservativeness parameter α are represented in Fig. 8. As α increases the divergence load increases until $\alpha=\alpha_2$, this can be regarded as the result of the lower bound theorem (5).

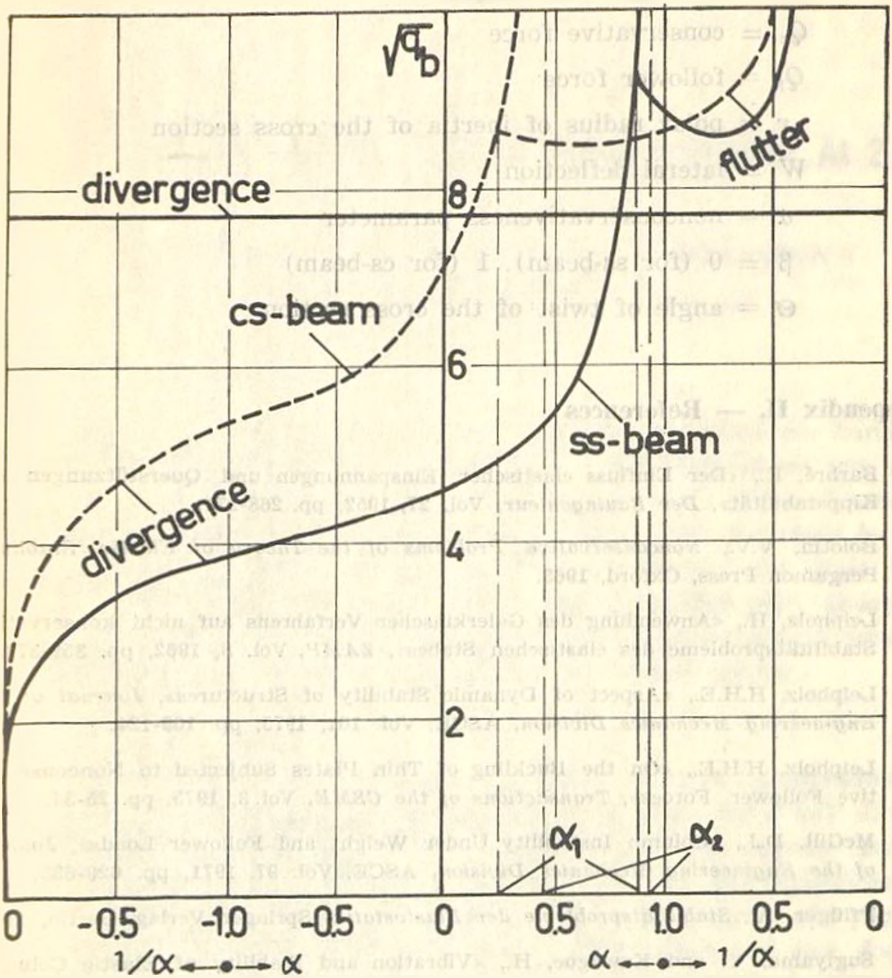

Fig. 8. Divergence and flutter loads of the ss- and cs-beams for $\lambda=0.1$ and $\mu=0.3$

Appendix I. — Notation

$EI(1-\mu^2)$ = small bending stiffness of the cross section

GJ = torsional stiffness of the cross section

h = height of the beam

l = length of the beam

m = mass per unit length

Q_c = conservative force

Q_f = follower force

r = polar radius of inertia of the cross section

W = lateral deflection

α = nonconservativeness parameter

β = 0 (for ss-beam), 1 (for cs-beam)

Θ = angle of twist of the cross section

Appendix II. — References

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