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Research article

A general method for solving linear matrix equations of elliptic biquaternions with applications

Kahraman Esen Özen*

Department of Mathematics, Sakarya University, Sakarya, TR-54187, Turkey

* **Correspondence:** Email: kahraman.ozen1@ogr.sakarya.edu.tr.

Abstract: In this study, we obtain the real representations of elliptic biquaternion matrices. Afterwards, with the aid of these representations, we develop a general method to solve the linear matrix equations over the elliptic biquaternion algebra. Also we apply this method to the well known matrix equations $X - AXB = C$ and $AX - XB = C$ over the elliptic biquaternion algebra. Then, we give some illustrative numerical examples to show how the aforementioned method and its results work. Furthermore, we provide numerical algorithms for all the problems considered in this paper. Elliptic biquaternions are generalized form of complex quaternions and so real quaternions. This relation is valid for their matrices, as well. Thus, the obtained results extend, generalize and complement some known results from the literature.

Keywords: elliptic biquaternion; matrices of elliptic biquaternions; matrix equation; solution, real representation
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1. Introduction

Throughout the paper, the following notations are used. The set of real numbers, complex numbers, elliptic numbers, elliptic biquaternions are denoted by $\mathbb{R}, \mathbb{C}, \mathbb{C}_p$, $H\mathbb{C}_p$, respectively. The set of all matrices on $\mathbb{R}(\mathbb{C}_p\, or H\mathbb{C}_p)$ are denoted by $M_{m\times n}(\mathbb{R})(M_{m\times n}(\mathbb{C}_p)\, or M_{m\times n}(H\mathbb{C}_p))$. For convenience, the set of all square matrices on $\mathbb{C}_p(\, or\, H\mathbb{C}_p)$ are denoted by $M_s(\mathbb{C}_p)(\, or\, M_s(H\mathbb{C}_p))$.

In 1843, Hamilton introduced the set of real quaternions [1], which can be represented as

$$H = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where the quaternionic units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the equalities:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\,, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}\,, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{1.1}$$

There are many applications of real quaternions in various areas of science. One of these applications is related to matrix theory. In the first half of the 20th century, the real quaternion matrices began to study [2]. It is well known that linear matrix equations with their applications have been one of the main topics in matrix theory. For real quaternion matrices, Tian discussed the linear equations in [3], and gave a general method to solve them. Also, the real quaternion matrix equation $X - AXF = C$ (Lyapunov equation) is studied by Song et al. [4]. On the other hand, in [5], Song and Chen studied the real quaternion matrix equation $XF - AX = C$ (Sylvester equation). In the process of preparing this paper, we are motivated by the aforementioned studies [3–5]. The studies [6–9] can be suggested as some different and qualified studies on quaternion matrix equations.

After the discovery of real quaternion algebra, Hamilton also introduced the complex quaternion algebra [10]. The set of complex quaternions is defined by

$$H_{\mathbb{C}} = \{Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k} : \; Q_0, Q_1, Q_2, Q_3 \in \mathbb{C}\}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ satisfy the same multiplication rules given in (1.1).

As well as real quaternions, complex quaternions have many applications in many areas of science and they have an important role to explain mathematical and physical events. In these applications of real and complex quaternions, their complex matrix representations have an important place. There can be found some interesting studies on complex matrices in [11–14].

Recently, the elliptic biquaternion algebra, which includes the complex quaternion algebra and real quaternion algebra as special cases, has been introduced. Various studies concerned with elliptic biquaternion algebra have been presented in the literature. We refer the readers to the papers [15–19].

This article is organized as follows. In section 2, we review elliptic numbers, elliptic matrices, elliptic biquaternions and elliptic biquaternion matrices. In Section 3, real representations of elliptic biquaternion matrices are obtained. In section 4, in view of these real representations, we develop a general method to study the solutions of linear matrix equations over the elliptic biquaternion algebra $H\mathbb{C}_p$. In section 5, we investigate the solutions of the elliptic biquaternion matrix equations $X - AXB = C$ and $AX - XB = C$ by means of this method. In section 6, we provide numerical algorithms for finding the solutions of problems which are discussed in the section 4 and section 5.

2. Preliminaries

The set of elliptic numbers is represented as

$$\mathbb{C}_p = \left\{x + Iy : \; x, y \in \mathbb{R}, \; I^2 = p < 0, \; p \in \mathbb{R}\right\}.$$

In this number system, addition and multiplication of any elliptic numbers $\omega = x_1 + Iy_1, \; \varsigma = x_2 + Iy_2 \in \mathbb{C}_p$ are defined as $\omega + \varsigma = (x_1 + Iy_1) + (x_2 + Iy_2) = (x_1 + x_2) + I(y_1 + y_2)$ and $\omega\varsigma = (x_1 + Iy_1)(x_2 + Iy_2) = (x_1x_2 + py_1y_2) + I(x_1y_2 + x_2y_1)$, respectively. As it is well known in the literature, $\mathbb{C}_p$ is a field under these two operations, [20]. The set of matrices, which includes $m \times n$ matrices with elliptic number entries, are investigated in [21]. In the set of $m \times n$ elliptic matrices $M_{m\times n}\left(\mathbb{C}_p\right)$, ordinary matrix addition and multiplication are defined. Also, the scalar multiplication is defined as $\lambda A = \lambda\left[a_{ij}\right] = \left[\lambda a_{ij}\right] \in M_{m\times n}\left(\mathbb{C}_p\right)$ where $\lambda \in \mathbb{C}_p$ and $A = \left[a_{ij}\right] \in M_{m\times n}\left(\mathbb{C}_p\right)$, [21].

The elliptic biquaternion algebra is a four dimensional vector space over the elliptic number field $\mathbb{C}_p$. It is expressed by

$$H\mathbb{C}_p = \left\{ Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \ : \ A_0, A_1, A_2, A_3 \in \mathbb{C}_p \right\}$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the quaternionic units which satisfy (1.1). Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $R = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k} \in H\mathbb{C}_p$ and $\lambda \in \mathbb{C}_p$ be given. Then, the operations of multiplication and addition are expressed as

$$\begin{aligned} QR = {} & [(A_0B_0) - (A_1B_1) - (A_2B_2) - (A_3B_3)] + [(A_0B_1) + (A_1B_0) + (A_2B_3) - (A_3B_2)]\,\mathbf{i} \\ & + [(A_0B_2) - (A_1B_3) + (A_2B_0) + (A_3B_1)]\,\mathbf{j} + [(A_0B_3) + (A_1B_2) - (A_2B_1) + (A_3B_0)]\,\mathbf{k} \end{aligned}$$

$$Q + R = (A_0 + B_0) + (A_1 + B_1)\,\mathbf{i} + (A_2 + B_2)\,\mathbf{j} + (A_3 + B_3)\,\mathbf{k}$$

while the operation of scalar multiplication is expressed as [15]

$$\lambda Q = (\lambda A_0) + (\lambda A_1)\,\mathbf{i} + (\lambda A_2)\,\mathbf{j} + (\lambda A_3)\,\mathbf{k}.$$

The set of all $m \times n$ type matrices with elliptic biquaternion entries is denoted by $M_{m\times n}\left(H\mathbb{C}_p\right)$. In this set, the ordinary matrix addition and multiplication are defined. Also, the scalar multiplication is defined as

$$QA = Q\left[a_{ij}\right] = \left[Qa_{ij}\right] \in M_{m\times n}\left(H\mathbb{C}_p\right)$$

where $A = \left[a_{ij}\right] \in M_{m\times n}\left(H\mathbb{C}_p\right)$ and $Q \in H\mathbb{C}_p$. There is a faithful relation between elliptic matrices and elliptic biquaternion matrices. Where $A_0,\ A_1,\ A_2,\ A_3 \in M_{m\times n}\left(\mathbb{C}_p\right)$, every elliptic biquaternion matrix $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in M_{m\times n}\left(H\mathbb{C}_p\right)$ has a $2m \times 2n$ elliptic representation

$$\xi(A) = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} IA_1 & -A_2 - \frac{1}{\sqrt{|p|}} IA_3 \\ A_2 - \frac{1}{\sqrt{|p|}} IA_3 & A_0 - \frac{1}{\sqrt{|p|}} IA_1 \end{bmatrix} \tag{2.1}$$

which is determined by means of the following linear isomorphism [22]

$$\begin{aligned} \xi \ : \quad M_{m\times n}\left(H\mathbb{C}_p\right) & \ \to \ M_{2m\times 2n}\left(\mathbb{C}_p\right) \\ A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} & \ \to \ \xi(A) = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} IA_1 & -A_2 - \frac{1}{\sqrt{|p|}} IA_3 \\ A_2 - \frac{1}{\sqrt{|p|}} IA_3 & A_0 - \frac{1}{\sqrt{|p|}} IA_1 \end{bmatrix}. \end{aligned}$$

We aim to obtain the real representations of elliptic biquaternion matrices. In the next section, the representation $\xi(A)$ will be written in a somewhat different form which is suitable for the purpose of us.

3. Real representations of elliptic biquaternion matrices

In this section, firstly, we get the real representations of elliptic matrices. Afterwards, we obtain the real representations of elliptic biquaternion matrices which will be useful for investigating the solutions of linear matrix equations over the elliptic biquaternion algebra $H\mathbb{C}_p$ in the next section.

By means of the study [23] which was presented by Yaglom in 1968, we know that in the case $I^2 = -q-rI(r^2-4q < 0)$, the transformation that is obtained by making the generalized complex number $c_1 + Id_1$ correspond to the ordinary complex number $c + id$, where $c = c_1 - \frac{r}{2}d_1$ and $d = \frac{d_1}{2}\sqrt{4q - r^2}$, is an isomorphism. As it is well known, $I^2 = p,\ p < 0$ for elliptic numbers. By taking into consideration this case, the restriction of this isomorphism is obtained as follows:

$$\begin{aligned} \varepsilon \ : \ \mathbb{C}_p \ &\to \ \mathbb{C} \\ c_1 + Id_1 \ &\to \ c_1 + id_1\sqrt{|p|} \end{aligned} \tag{3.1}$$

On the other hand, we know that an ordinary complex number $z = a + ib$ has a faithful real matrix representation

$$\alpha(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

which is determined by means of the following linear isomorphism

$$\begin{aligned} \alpha \ : \ \mathbb{C} \to M_2^{\Omega}(\mathbb{R}) \\ z = a + ib \to \alpha(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \end{aligned} \tag{3.2}$$

where $M_2^{\Omega}(\mathbb{R}) = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : \ x, y \in \mathbb{R} \right\}$, [24].

Now, we take into consideration the compound function $\delta_p = \alpha \circ \varepsilon$ which is given as

$$\begin{aligned} \delta_p \ : \ \mathbb{C}_p \ &\to \ M_2^{\Omega}(\mathbb{R}) \\ \omega = x + Iy \ &\to \ \delta_p(\omega) = \begin{bmatrix} x & -y\sqrt{|p|} \\ y\sqrt{|p|} & x \end{bmatrix}. \end{aligned} \tag{3.3}$$

Since the functions ε and α are linear isomorphisms, there is no doubt that δ_p is a linear isomorphism. Thus, we have a faithful real matrix representation of an elliptic number $\omega = x + Iy \in \mathbb{C}_p$ as

$$\delta_p(\omega) = \begin{bmatrix} x & -y\sqrt{|p|} \\ y\sqrt{|p|} & x \end{bmatrix}.$$

As a natural consequence of the linear isomorphism δ_p, the following function

$$\begin{aligned} \gamma_p \ : \ M_{m\times n}(\mathbb{C}_p) \ &\to \ M_{2m\times 2n}^{\Omega}(\mathbb{R}) \\ A = A_1 + IA_2 \ &\to \ \gamma_p(A) = \begin{bmatrix} A_1 & -\sqrt{|p|}A_2 \\ \sqrt{|p|}A_2 & A_1 \end{bmatrix} \end{aligned}$$

that is expected to be a linear isomorphism can be immediately defined where $M^{\Omega}_{2m\times 2n}(\mathbb{R}) = \left\{ \begin{bmatrix} G & -\sqrt{|p|}H \\ \sqrt{|p|}H & G \end{bmatrix} : G, H \in M_{m\times n}(\mathbb{R}) \right\}$. One can see easily that this function is bijection and satisfies the following equalities

$$\gamma_p(A + B) = \gamma_p(A) + \gamma_p(B), \quad \gamma_p(AC) = \gamma_p(A)\gamma_p(C)$$

for any elliptic matrices A, B, C of appropriate sizes. Thus, γ_p is a linear isomorphism as anticipated and we have a faithful real matrix representation of an elliptic matrix $A = A_1 + IA_2 \in M_{m\times n}(\mathbb{C}_p)$ as

$$\gamma_p(A) = \begin{bmatrix} A_1 & -\sqrt{|p|}A_2 \\ \sqrt{|p|}A_2 & A_1 \end{bmatrix}. \tag{3.4}$$

On the other hand, the elliptic matrix representation of an elliptic biquaternion matrix $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in M_{m\times n}(H\mathbb{C}_p)$ given in (2.1) can be written in a somewhat different form as

$$\xi(A) = \begin{bmatrix} A_0^{\#} - \sqrt{|p|}A_1' & -A_2^{\#} + \sqrt{|p|}A_3' \\ A_2^{\#} + \sqrt{|p|}A_3' & A_0^{\#} + \sqrt{|p|}A_1' \end{bmatrix} + I \begin{bmatrix} A_0' + \frac{1}{\sqrt{|p|}}A_1^{\#} & -A_2' - \frac{1}{\sqrt{|p|}}A_3^{\#} \\ A_2' - \frac{1}{\sqrt{|p|}}A_3^{\#} & A_0' - \frac{1}{\sqrt{|p|}}A_1^{\#} \end{bmatrix} \tag{3.5}$$

where $A_i = A_i^{\#} + I\,A_i' \in M_{m\times n}(\mathbb{C}_p)$, $A_i^{\#}, A_i' \in M_{m\times n}(\mathbb{R})$, $0 \leq i \leq 3$. Then, applying (3.4) to (3.5), we get $4m \times 4n$ real matrix representation of the elliptic matrix $\xi(A)$ (in other words the real representation of the elliptic biquaternion matrix A) as follows:

$$\gamma_p(\xi(A)) = \begin{bmatrix} A_0^{\#} - \sqrt{|p|}A_1' & -A_2^{\#} + \sqrt{|p|}A_3' & -A_1^{\#} - \sqrt{|p|}A_0' & A_3^{\#} + \sqrt{|p|}A_2' \\ A_2^{\#} + \sqrt{|p|}A_3' & A_0^{\#} + \sqrt{|p|}A_1' & A_3^{\#} - \sqrt{|p|}A_2' & A_1^{\#} - \sqrt{|p|}A_0' \\ A_1^{\#} + \sqrt{|p|}A_0' & -A_3^{\#} - \sqrt{|p|}A_2' & A_0^{\#} - \sqrt{|p|}A_1' & -A_2^{\#} + \sqrt{|p|}A_3' \\ -A_3^{\#} + \sqrt{|p|}A_2' & -A_1^{\#} + \sqrt{|p|}A_0' & A_2^{\#} + \sqrt{|p|}A_3' & A_0^{\#} + \sqrt{|p|}A_1' \end{bmatrix}. \tag{3.6}$$

For convenience, let us denote $\gamma_p(\xi(A))$ by $(A)_{\gamma p}$ where A is any elliptic biquaternion matrix. One can immediately see that the unit matrix I_n satisfies the equation

$$(I_n)_{\gamma p} = I_{4n}. \tag{3.7}$$

Some properties which are satisfied by the real representation are given below.

Proposition 3.1. *Let $A, B \in M_{m\times n}(H\mathbb{C}_p)$, $C \in M_{n\times l}(H\mathbb{C}_p)$, $D \in M_s(H\mathbb{C}_p)$ be arbitrary elliptic biquaternion matrices. In that case*

1. $A = B \Leftrightarrow (A)_{\gamma p} = (B)_{\gamma p}$,
2. $(A + B)_{\gamma p} = (A)_{\gamma p} + (B)_{\gamma p}$, $(AC)_{\gamma p} = (A)_{\gamma p}(C)_{\gamma p}$,

3. *If D is invertible, then* $(D)_{\gamma p}$ *is invertible and* $(D^{-1})_{\gamma p} = ((D)_{\gamma p})^{-1}$*,*

4. $(A)_{\gamma p} = S_{4m}{}^{-1}(A)_{\gamma p}S_{4n}$ *where* $S_{4t} = \begin{bmatrix} 0 & 0 & -I_t & 0 \\ 0 & 0 & 0 & -I_t \\ I_t & 0 & 0 & 0 \\ 0 & I_t & 0 & 0 \end{bmatrix}, \ t = m, n.$

Proof. Since $\gamma_p \circ \varepsilon$ is a linear isomorphism, 1 and 2 are obvious. Also, the proof of 4 can be easily completed by direct calculation. Now we will prove 3.

3. From the inverse property, we can write

$$DD^{-1} = D^{-1}D = I_s.$$

Then, we get the equalities

$$(D)_{\gamma p}(D^{-1})_{\gamma p} = (DD^{-1})_{\gamma p} = (I_s)_{\gamma p} = I_{4s}$$

and

$$(D^{-1})_{\gamma p}(D)_{\gamma p} = (D^{-1}D)_{\gamma p} = (I_s)_{\gamma p} = I_{4s}$$

by means of (3.7) and first two properties in this proposition. It means that $(D^{-1})_{\gamma p} = ((D)_{\gamma p})^{-1}$.

□

4. On the solutions of linear matrix equations over $H\mathbb{C}_p$

In this section, we give a general method on the solutions of linear matrix equations over the elliptic biquaternion algebra $H\mathbb{C}_p$ with the aid of the real representations. To do so, we take into consideration the elliptic biquaternion matrix equation:

$$A_1XB_1 + ... + A_kXB_k = C \tag{4.1}$$

where $A_1, ..., A_k \in M_{m\times n}(H\mathbb{C}_p)$, $B_1, ..., B_k \in M_{u\times v}(H\mathbb{C}_p)$, $C \in M_{m\times v}(H\mathbb{C}_p)$ and $X \in M_{n\times u}(H\mathbb{C}_p)$.

Let us define the real representation of the elliptic biquaternion matrix equation (4.1) as in the following:

$$(A_1)_{\gamma p}Y(B_1)_{\gamma p} + ... + (A_k)_{\gamma p}Y(B_k)_{\gamma p} = (C)_{\gamma p}. \tag{4.2}$$

Thanks to the first two properties given in Proposition 3.1, the elliptic biquaternion matrix equation (4.1) is equivalent to the following real matrix equation

$$(A_1)_{\gamma p}(X)_{\gamma p}(B_1)_{\gamma p} + ... + (A_k)_{\gamma p}(X)_{\gamma p}(B_k)_{\gamma p} = (C)_{\gamma p}. \tag{4.3}$$

Hence, we have the following proposition.

Proposition 4.1. *The elliptic biquaternion matrix equation* (4.1) *has an elliptic biquaternion matrix solution* $X \in M_{n\times u}(H\mathbb{C}_p)$ *if and only if the real matrix equation* (4.2) *has a real matrix solution* $Y = (X)_{\gamma p} \in M_{4n\times 4u}(\mathbb{R})$.

Theorem 4.2. *Let $A_1, A_2, ..., A_k \in M_{m\times n}(H\mathbb{C}_p)$, $B_1, B_2, ..., B_k \in M_{u\times v}(H\mathbb{C}_p)$ and $C \in M_{m\times v}(H\mathbb{C}_p)$ be given. In this case the elliptic biquaternion matrix equation* (4.1) *has a solution $X \in M_{n\times u}(H\mathbb{C}_p)$ if and only if its real representation equation* (4.2) *has a solution $Y \in M_{4n\times 4u}(\mathbb{R})$. In that case, if the block matrix*

$$Y = \left[Y_{ij}\right]_{i,j=1}^{4}, \quad Y_{ij} \in M_{n\times u}(\mathbb{R})$$

is a solution of (4.2)*, then $n \times u$ elliptic biquaternion matrix*

$$X = (X_0^{\#} + IX_0') + (X_1^{\#} + IX_1')\mathbf{i} + (X_2^{\#} + IX_2')\mathbf{j} + (X_3^{\#} + IX_3')\mathbf{k} \tag{4.4}$$

is a solution of (4.1) *where*

$$\begin{aligned} X_0^{\#} &= \tfrac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}), \;\; X_0' = \tfrac{1}{4\sqrt{|p|}}(Y_{31} - Y_{13} + Y_{42} - Y_{24}), \\ X_1^{\#} &= \tfrac{1}{4}(Y_{24} - Y_{42} + Y_{31} - Y_{13}), \;\; X_1' = \tfrac{1}{4\sqrt{|p|}}(Y_{44} - Y_{33} + Y_{22} - Y_{11}), \\ X_2^{\#} &= \tfrac{1}{4}(Y_{21} - Y_{12} + Y_{43} - Y_{34}), \;\; X_2' = \tfrac{1}{4\sqrt{|p|}}(Y_{14} - Y_{32} + Y_{41} - Y_{23}), \\ X_3^{\#} &= \tfrac{1}{4}(Y_{14} - Y_{32} + Y_{23} - Y_{41}), \;\; X_3' = \tfrac{1}{4\sqrt{|p|}}(Y_{34} + Y_{12} + Y_{43} + Y_{21}). \end{aligned} \tag{4.5}$$

Proof. In view of the Proposition 4.1, the proof remains to show that if the block real matrix

$$Y = \left[Y_{ij}\right]_{i,j=1}^{4}, \quad Y_{ij} \in M_{n\times u}(\mathbb{R}) \tag{4.6}$$

is a solution of (4.2), in that case the elliptic biquaternion matrix, which is given in (4.4), is a solution of (4.1). When Y is a solution of (4.2), in view of the fourth property in Proposition 3.1, we have the equation

$$(A_1)_{\gamma p}(S_{4n}{}^{-1}YS_{4u})(B_1)_{\gamma p} + ... + (A_k)_{\gamma p}(S_{4n}{}^{-1}YS_{4u})(B_k)_{\gamma p} = (C)_{\gamma p}. \tag{4.7}$$

This last equation shows that $S_{4n}{}^{-1}YS_{4u}$ is also a solution of (4.2). Then, according to the matrix theory, the following matrix

$$Y' = \frac{1}{2}(Y + S_{4n}{}^{-1}YS_{4u}) \tag{4.8}$$

satisfies the real matrix equation (4.2), that is, Y' is another solution of (4.2). If (4.6) is substituted in (4.8), after some calculations, the equality

$$Y' = \begin{bmatrix} J & K & -N & -O \\ L & M & -P & -R \\ N & O & J & K \\ P & R & L & M \end{bmatrix} \tag{4.9}$$

is obtained where

$$J = \frac{1}{2}(Y_{11} + Y_{33}),\ L = \frac{1}{2}(Y_{21} + Y_{43}),\ N = \frac{1}{2}(Y_{31} - Y_{13}),\ P = \frac{1}{2}(Y_{41} - Y_{23})$$

$$K = \frac{1}{2}(Y_{12} + Y_{34}),\ M = \frac{1}{2}(Y_{22} + Y_{44}),\ O = \frac{1}{2}(Y_{32} - Y_{14}),\ R = \frac{1}{2}(Y_{42} - Y_{24}).$$

By taking into consideration (4.9) and (3.6), we construct the elliptic biquaternion matrix

$$X = (X_0^{\#} + I\,X_0') + (X_1^{\#} + I\,X_1')\mathbf{i} + (X_2^{\#} + I\,X_2')\mathbf{j} + (X_3^{\#} + I\,X_3')\mathbf{k}$$

where $X_i^{\#}$, X_i', $0 \leq i \leq 3$ are as in (4.5).

Obviously, $(X)_{\gamma p} = Y'$. Then, according to Proposition 4.1, the elliptic biquaternion matrix X is a solution of the equation (4.1). □

5. Some results

In this section, we investigate the solutions of the elliptic biquaternion matrix equations $X - AXB = C$ and $AX - XB = C$ by means of Theorem 4.2.

For $k = 2$, the special case of (4.1) is given by

$$A_1XB_1 + A_2XB_2 = C \tag{5.1}$$

where $A_1, A_2 \in M_{m\times n}(H\mathbb{C}_p)$, $B_1, B_2 \in M_{u\times v}(H\mathbb{C}_p)$, $C \in M_{m\times v}(H\mathbb{C}_p)$ and $X \in M_{n\times u}(H\mathbb{C}_p)$. If $B_1 = I_u$, $A_2 = -I_n$, $m = n$, $u = v$ are taken in (5.1) and also some notation changes are made as follows: $A_1 = A$, $B_2 = B$,

$$AX - XB = C$$

is obtained where $A \in M_n(H\mathbb{C}_p)$, $B \in M_u(H\mathbb{C}_p)$ and $C \in M_{n\times u}(H\mathbb{C}_p)$. It is not difficult to see that the real representation of the last equation is

$$(A)_{\gamma p}Y - Y(B)_{\gamma p} = (C)_{\gamma p}.$$

In view of the above derivation and Theorem 4.2, we have the following corollary:

Corollary 5.1. *Let $A \in M_n(H\mathbb{C}_p)$, $B \in M_u(H\mathbb{C}_p)$ and $C \in M_{n\times u}(H\mathbb{C}_p)$. In this case the elliptic biquaternion matrix equation*

$$AX - XB = C \tag{5.2}$$

has a solution $X \in M_{n\times u}(H\mathbb{C}_p)$ if and only if the real matrix equation

$$(A)_{\gamma p}Y - Y(B)_{\gamma p} = (C)_{\gamma p} \tag{5.3}$$

has a solution $Y \in M_{4n\times 4u}(\mathbb{R})$*, in which case, if*

$$Y = \left[Y_{ij}\right]_{i,j=1}^{4}, \quad Y_{ij} \in M_{n\times u}(\mathbb{R})$$

is a solution of (5.3)*, then* $n \times u$ *elliptic biquaternion matrix*

$$X = (X_0^{\#} + IX_0') + (X_1^{\#} + IX_1')\mathbf{i} + (X_2^{\#} + IX_2')\mathbf{j} + (X_3^{\#} + IX_3')\mathbf{k}$$

is a solution of (5.2) *where* $X_i^{\#}$, X_i', $0 \le i \le 3$ *are calculated as in* (4.5). □

Similarly above, if $A_1 = I_n$, $B_1 = I_u$, $m = n$, $u = v$ are taken in (5.1) and also some notation changes are made as follows: $A_2 = -A$, $B_2 = B$,

$$X - AXB = C$$

is obtained where $A \in M_n(H\mathbb{C}_p)$, $B \in M_u(H\mathbb{C}_p)$ and $C \in M_{n\times u}(H\mathbb{C}_p)$. It is easy to see that the real representation of the last equation is

$$Y - (A)_{\gamma p} Y (B)_{\gamma p} = (C)_{\gamma p}.$$

Considering the above derivation and Theorem 4.2, we have the following corollary:

Corollary 5.2. *Let* $A \in M_n(H\mathbb{C}_p)$, $B \in M_u(H\mathbb{C}_p)$ *and* $C \in M_{n\times u}(H\mathbb{C}_p)$. *In this case the elliptic biquaternion matrix equation*

$$X - AXB = C \tag{5.4}$$

has a solution $X \in M_{n\times u}(H\mathbb{C}_p)$ *if and only if the real matrix equation*

$$Y - (A)_{\gamma p} Y (B)_{\gamma p} = (C)_{\gamma p} \tag{5.5}$$

has a solution $Y \in M_{4n\times 4u}(\mathbb{R})$*, in which case, if*

$$Y = \left[Y_{ij}\right]_{i,j=1}^{4}, \quad Y_{ij} \in M_{n\times u}(\mathbb{R})$$

is a solution of (5.5)*, then* $n \times u$ *elliptic biquaternion matrix*

$$X = (X_0^{\#} + IX_0') + (X_1^{\#} + IX_1')\mathbf{i} + (X_2^{\#} + IX_2')\mathbf{j} + (X_3^{\#} + IX_3')\mathbf{k}$$

is a solution of (5.4) *where* $X_i^{\#}$, X_i', $0 \le i \le 3$ *are calculated as in* (4.5).

6. Numerical algorithms and examples

Based on the discussions in Section 4 and Section 5, in this section we provide numerical algorithms for finding the solutions of problems which are related to Corollary 5.1, Corollary 5.2 and Theorem 4.2.

Note that all computations in the rest of the paper are performed on an Intel i7-3630QM@2.40 Ghz/16GB computer using MATLAB R2016a software. Another thing that can be of importance is that we use the standard MATLAB package procedures.

Firstly, we give an example related to Corollary 5.1.

Example 6.1. Solve elliptic biquaternion matrix equation

$$\begin{bmatrix} -(1+I)\mathbf{i} & (2I)\,\mathbf{j} \\ 0 & 0 \end{bmatrix} X - X \begin{bmatrix} 1 & \mathbf{i} \\ 0 & \mathbf{k} \end{bmatrix}$$

$$= \begin{bmatrix} (-8+2I)-(1+I)\,\mathbf{i}+(-18+2I)\,\mathbf{j}-2I\mathbf{k} & (19-3I)-(2+I)\,\mathbf{i}-2I\mathbf{j}+(25-3I)\,\mathbf{k} \\ 0 & -(2+I)\,\mathbf{i}+I\mathbf{j} \end{bmatrix}$$

over the elliptic biquaternion algebra $H\mathbb{C}_{-9}$ with the aid of its real representation.

By taking into consideration the Corollary 5.1, real representation of given matrix equation can be written as in the following:

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & -6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -6 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} Y - Y \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 22 & 12 & -9 & -5 & 11 & 6 & 19 \\ 0 & 3 & 0 & 0 & 0 & 2 & 0 & 3 \\ -24 & -9 & -11 & 16 & -5 & -1 & -6 & 31 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & -3 \\ 5 & -11 & -6 & -19 & -5 & 22 & 12 & -9 \\ 0 & -2 & 0 & -3 & 0 & 3 & 0 & 0 \\ 6 & -31 & 7 & -7 & -24 & -9 & -11 & 16 \\ 0 & 3 & 0 & 2 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

If we solve this equation, we have

$$Y = \begin{bmatrix} -3 & 0 & 6 & -2 & -1 & 0 & 0 & 3 \\ 0 & -3 & 0 & -2 & 0 & 0 & 0 & 3 \\ 6 & 2 & 3 & 0 & 0 & -3 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -3 & -3 & 0 & 6 & -2 \\ 0 & 0 & 0 & -3 & 0 & -3 & 0 & -2 \\ 0 & 3 & -1 & 0 & 6 & 2 & 3 & 0 \\ 0 & 3 & 0 & 0 & 0 & 2 & 0 & 3 \end{bmatrix}.$$

It means that

$$Y_{11}=Y_{33}=\begin{bmatrix}-3&0\\0&-3\end{bmatrix}, Y_{12}=Y_{34}=\begin{bmatrix}6&-2\\0&-2\end{bmatrix}, Y_{14}=Y_{41}=\begin{bmatrix}0&3\\0&3\end{bmatrix}, Y_{13}=Y_{42}=\begin{bmatrix}-1&0\\0&0\end{bmatrix},$$

$$Y_{22}=Y_{44}=\begin{bmatrix}3&0\\0&3\end{bmatrix},\ Y_{24}=Y_{31}=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ Y_{32}=Y_{23}=\begin{bmatrix}0&-3\\0&-3\end{bmatrix},\ Y_{21}=Y_{43}=\begin{bmatrix}6&2\\0&2\end{bmatrix}.$$

Consequently, it is concluded that

$$X=\begin{bmatrix}(1+I)\,\mathbf{i}+2I\mathbf{k} & (2+I)\,\mathbf{j}\\ 0 & I\mathbf{i}+(2+I)\,\mathbf{j}\end{bmatrix}\in M_2(H\mathbb{C}_{-9}).$$

by means of the equations (4.4) and (4.5). □

We can generate an algorithm for problems related to Corollary 5.1 as in the following:

Algorithm 1

1. Input A, B, C ($A\in M_n(H\mathbb{C}_p)$, $B\in M_u(H\mathbb{C}_p)$ and $C\in M_{n\times u}(H\mathbb{C}_p)$).
2. Form $(A)_{\gamma p}$, $(B)_{\gamma p}$, $(C)_{\gamma p}$.
3. Compute $Y=\left[Y_{ij}\right]_{i,j=1}^{4}$ satisfying (5.3) ($Y_{ij}\in M_{n\times u}(\mathbb{R})$).
4. Calculate $X_i^{\#}$, X_i' according to (4.5)($0\le i\le 3$).
5. Output $X=(X_0^{\#}+IX_0')+(X_1^{\#}+IX_1')\mathbf{i}+(X_2^{\#}+IX_2')\mathbf{j}+(X_3^{\#}+I\,X_3')\mathbf{k}$.

Now, we give an example related to Corollary 5.2.

Example 6.2. Solve elliptic biquaternion matrix equation

$$X-\begin{bmatrix}(-1-I)\mathbf{i}&0\\0&0\end{bmatrix}X\begin{bmatrix}1&0\\0&\mathbf{k}\end{bmatrix}$$

$$=\begin{bmatrix}(1-I)+(1+I)\,\mathbf{i}+(I)\,\mathbf{j}+(-1+2I)\,\mathbf{k} & (1-2I)\,\mathbf{i}+(1+I)\,\mathbf{j}+(1+I)\,\mathbf{k}\\ 0 & (I)\,\mathbf{i}+5\mathbf{j}\end{bmatrix}$$

over the elliptic biquaternion algebra $H\mathbb{C}_{-2}$ by using its real representation.

If the Corollary 5.2 is taken into consideration, the real representation equation

$$Y-\begin{bmatrix}1.4142&0&-1&0&1&0&0&0\\0&0&0&0&0&0&0&0\\1&0&-1.4142&0&0&0&-1&0\\0&0&0&0&0&0&0&0\\-1&0&0&0&1.4142&0&-1&0\\0&0&0&0&0&0&0&0\\0&0&1&0&1&0&-1.4142&0\\0&0&0&0&0&0&0&0\end{bmatrix}Y\begin{bmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&-1&0&0&0&0&0&0\end{bmatrix}$$

$$
= \begin{bmatrix}
-0.4142 & 2.8284 & 2.8284 & 0.4142 & 0.4142 & -1 & 0.4142 & 2.4142 \\
0 & -1.4142 & 0 & -5 & 0 & 0 & 0 & 0 \\
2.8284 & 2.4142 & 2.4142 & -2.8284 & -2.4142 & -0.4142 & 2.4142 & 1 \\
0 & 5 & 0 & 1.4142 & 0 & 0 & 0 & 0 \\
-0.4142 & 1 & -0.4142 & -2.4142 & -0.4142 & 2.8284 & 2.8284 & 0.4142 \\
0 & 0 & 0 & 0 & 0 & -1.4142 & 0 & -5 \\
2.4142 & 0.4142 & -2.4142 & -1 & 2.8284 & 2.4142 & 2.4142 & -2.8284 \\
0 & 0 & 0 & 0 & 0 & 5 & 0 & 1.4142
\end{bmatrix}
$$

can be written. By solving this equation, we have

$$
Y = \begin{bmatrix}
-1.4142 & 0 & 0 & 1.4142 & -1 & 0 & 1.4142 & 1 \\
0 & -1.4142 & 0 & -5 & 0 & 0 & 0 & 0 \\
0 & 1.4142 & 1.4142 & 0 & -1.4142 & 1 & 1 & 0 \\
0 & 5 & 0 & 1.4142 & 0 & 0 & 0 & 0 \\
1 & 0 & -1.4142 & -1 & -1.4142 & 0 & 0 & 1.4142 \\
0 & 0 & 0 & 0 & 0 & -1.4142 & 0 & -5 \\
1.4142 & -1 & -1 & 0 & 0 & 1.4142 & 1.4142 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & 0 & 1.4142
\end{bmatrix}.
$$

It means that

$$
Y_{11} = Y_{33} = \begin{bmatrix} -1.4142 & 0 \\ 0 & -1.4142 \end{bmatrix},\ Y_{12} = Y_{34} = \begin{bmatrix} 0 & 1.4142 \\ 0 & -5 \end{bmatrix},\ Y_{14} = \begin{bmatrix} 1.4142 & 1 \\ 0 & 0 \end{bmatrix},
$$

$$
Y_{13} = Y_{42} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},\ Y_{21} = Y_{43} = \begin{bmatrix} 0 & 1.4142 \\ 0 & 5 \end{bmatrix},\ Y_{23} = \begin{bmatrix} -1.4142 & 1 \\ 0 & 0 \end{bmatrix},\ Y_{41} = \begin{bmatrix} 1.4142 & -1 \\ 0 & 0 \end{bmatrix},
$$

$$
Y_{22} = Y_{44} = \begin{bmatrix} 1.4142 & 0 \\ 0 & 1.4142 \end{bmatrix},\ Y_{24} = Y_{31} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ Y_{32} = \begin{bmatrix} -1.4142 & -1 \\ 0 & 0 \end{bmatrix}.
$$

Consequently, we obtain

$$
X = \begin{bmatrix} (1 + 0.9995I)\,\mathbf{i} + (0.9995I)\,\mathbf{j} & (1 + 0.9995I)\,\mathbf{k} \\ 0 & (0.9995I)\,\mathbf{i} + 5\mathbf{j} \end{bmatrix} \in M_2(H\mathbb{C}_{-2})
$$

by means of the equations (4.4) and (4.5). □

We can generate an algorithm for problems related to Corollary 5.2 as follows:

Algorithm 2

1. Input A, B, C ($A \in M_n(H\mathbb{C}_p)$, $B \in M_u(H\mathbb{C}_p)$ and $C \in M_{n\times u}(H\mathbb{C}_p)$).
2. Form $(A)_{\gamma p}$, $(B)_{\gamma p}$, $(C)_{\gamma p}$.
3. Compute $Y = \left[Y_{ij}\right]_{i,j=1}^{4}$ satisfying (5.5) ($Y_{ij} \in M_{n\times u}(\mathbb{R})$).
4. Calculate $X_i^{\#}$, X_i' according to (4.5)($0 \leq i \leq 3$).
5. Output $X = (X_0^{\#} + IX_0') + (X_1^{\#} + IX_1')\mathbf{i} + (X_2^{\#} + IX_2')\mathbf{j} + (X_3^{\#} + I\,X_3')\mathbf{k}$.

As a result of using the standard MATLAB package procedures, when our calculations include the rational numbers, root numbers, exponential expressions, logarithmic expressions etc., our method gives an approximate solution of the desired equation just like in the case of Example 6.2. Otherwise, our method gives the exact solution of the desired equation just like in the case of Example 6.1. To ensure the exact solution of the desired equation in Example 6.2, one can use the MATLAB package Symbolic Math Toolbox. If the same steps are followed by using this package, the exact solution

$$X = \begin{bmatrix} (1+I)\,\mathbf{i} + (I)\,\mathbf{j} & (1+I)\,\mathbf{k} \\ 0 & (I)\,\mathbf{i} + 5\mathbf{j} \end{bmatrix} \in M_2(H\mathbb{C}_{-2})$$

of the aforementioned elliptic biquaternion matrix equation is immediately found. It must be noted that using this package always provides an advantage in terms of the exact solution, however using it sometimes causes a disadvantage in terms of the length of the solution.

Finally, we can give an algorithm for the most general case, that is, for the problems related to Theorem 4.2.

Algorithm 3

1. Input A_i, B_i, C ($A_i \in M_{m\times n}(H\mathbb{C}_p)$, $B_i \in M_{u\times v}(H\mathbb{C}_p)$, $1 \leq i \leq k$ and $C \in M_{m\times v}(H\mathbb{C}_p)$).
2. Form $(A_i)_{\gamma p}$, $(B_i)_{\gamma p}$, $(C)_{\gamma p}$ ($1 \leq i \leq k$).
3. Compute $Y = \left[Y_{ij}\right]_{i,j=1}^{4}$ satisfying (4.2) ($Y_{ij} \in M_{n\times u}(\mathbb{R})$).
4. Calculate $X_i^{\#}$, X_i' according to (4.5)($0 \leq i \leq 3$).
5. Output $X = (X_0^{\#} + IX_0') + (X_1^{\#} + IX_1')\mathbf{i} + (X_2^{\#} + IX_2')\mathbf{j} + (X_3^{\#} + I\,X_3')\mathbf{k}$.

7. Conclusion

In this paper, real representations of elliptic biquaternion matrices, which may be needed to investigate various topics on elliptic biquaternion matrices in the future, are obtained. By means of these representations, a general method on the solutions of linear matrix equations over the elliptic biquaternion algebra $H\mathbb{C}_p$ is developed. Also, some problems are considered as applications of this method. Lastly, the numerical algorithms for finding the solutions of these problems are provided.

When $p = -1$ the number system $\mathbb{C}_p$ and the set of elliptic biquaternions $H\mathbb{C}_p$ correspond to the complex number system $\mathbb{C}$ and the set of complex quaternions $H_{\mathbb{C}}$, respectively. As a natural consequence of this case, the set of elliptic biquaternion matrices $M_{m\times n}(H\mathbb{C}_p)$ is reduced to set of complex quaternion matrices $M_{m\times n}(H_{\mathbb{C}})$ when $p = -1$. Therefore, our method solves the linear equations of complex quaternion matrices as well.

Real or complex quaternion matrices have an important role in many areas of science. Since elliptic biquaternion matrices are generalized form of complex quaternion matrices and so real quaternion matrices, it is expected that the results obtained here will be used as a valuable tool in many areas of science.

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Conflict of interest

The author declares that there is no conflict of interest.

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