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# Compactness of soft cone metric space and fixed point theorems related to diametrically contractive mapping

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**Abstract:** In this article, we describe the concepts such as sequentially soft closeness, sequential compactness, totally boundedness and sequentially continuity in any soft cone metric space and prove their some properties. Also, we examine soft closed set, soft closure, compactness and continuity in an elementary soft topological cone metric space. Unlike classical cone metric space, sequential compactness and compactness are not the same here. Because the compactness is an elementary soft topological property and cannot be defined for every soft cone metric space. However, in the restricted soft cone metric spaces, they are the same. Additionally, we prove some fixed point theorems related to diametrically contractive mapping in a complete soft cone metric space.

**Key words:** Elementary soft topology, soft cone metric, soft net, soft cover, soft compactness, soft fixed point

## 1. Introduction

Molodtsov [32] introduced the soft set theory in 1999. He applied this theory to solve problems in medical science, economics, social science, etc. Over time, many researchers have become interested in soft set and their applications. Maji et al. [28, 29] examined the soft set theory at great length and also applied soft sets to decision making problems. After these studies, important works were done in this direction [13, 27, 38, 45]. Some researchers [6, 40] have established the topologies on soft sets and have viewed topological properties of soft sets. Samanta and Das gave the concepts of soft element, soft real numbers [17] and soft complex numbers [18] on soft sets. Samanta et al. [16, 19, 30] studied mathematical structures such as soft vector, soft norm, soft metric etc. Das and Samanta [20] introduced the soft metric concept via the notation of soft element and proved Banach fixed point theorem. Taşköprü and Altıntaş [42] established a new topological structure on soft sets using elementary operations. Nowadays, many researches make interesting studies by applying the soft set theory to almost every area. For some of those, see [23, 33–37].

Other than this, Huang and Zhang [24] presented the theory of cone metric space with the normal cone as a generalisation of metric spaces. Then, Hamlbarani and Rezapour [39] proved the consequences of [24] without having to take into account the normality of cone. In recent years, a lot of researchers viewed the fixed point theory in the generalisations of the metric space ([1, 2, 5, 7–12, 14, 15, 21, 22, 25, 26, 31, 43, 44] and others).

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Abbas et al. [3] proved several important fixed point theorems in soft metric spaces via the soft point. Şimşek et al. [41] introduced soft cone metric structure using the soft element notation that are different from the soft points and proved several fixed point theorems and Altıntaş et al. [4] studied the elementary soft topology of soft cone metric spaces.

Here, we examine sequential compactness and compactness of soft cone metric spaces. We are interested in fixed point theory for diametrically contractive mapping in a complete soft cone metric space. Altıntaş et al. [4] showed that all soft cone metric spaces are not an elementary soft topological spaces. However, with some restriction [see $(D4)$], a soft cone metric space becomes an elementary soft topological space. Thus, the sequential compactness that can be defined in the soft cone metric space and compactness that can be defined only in elementary soft topological cone metric space. Remember that these two concepts are not the same in any classical cone metric spaces. Here, we work the concepts such as soft closeness, soft closure, sequentially soft closeness, soft net, sequential compactness, totally boundedness, soft compactness, first countability, continuity and diametrically contractive mapping and asymptotically diametrically contractive mapping.

## 2. Preliminary information

**Definition 2.1** *[32] Assume that $P$ is a parameters set, $U$ is a universal set and $P(U)$ is the power set of $U$. We call a pair $(G,P)$ a soft set on $U$ if $G:P\to P(U)$ is a mapping. That is, a soft set on $U$ is a parametrized class of subsets of $U$. We can regard a soft set as the $\alpha-$ approximate elements set of $(G,P)$ for each $\alpha\in P$.*

*We call a function $\varepsilon:P\to U$ a soft element of $U$. If for each $\alpha\in P$, $\varepsilon(\alpha)\in G(\alpha)$, $\varepsilon$ belongs to $(G,P)$. If $G(\alpha)$ is a singleton set for each $\alpha\in P$, then the soft set $(G,P)$ itself can be taken as a soft element. The soft elements are denoted $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, etc. and the class of soft elements of $(G,P)$ by $SE(G,P)$. For details, see [20].*

**Definition 2.2** *[20] Assume that $P$ is a parameters set and $B(\mathbb{R})$ is the class of nonempty all bounded subsets of real numbers set $\mathbb{R}$. We call a mapping $G:P\to B(\mathbb{R})$ a soft real set and denote by $(G,P)$. If $G(\alpha)$ is a singleton set for each $\alpha\in P$, then $(G,P)$ is a soft real number. Soft real numbers are denoted $\tilde{s}$, $\tilde{t}$, etc. Specially, for each $\alpha\in P$, a soft real number satisfying $\tilde{s}(\alpha)=s$, is denoted by $\bar{s}$. We denote all the soft real numbers class and nonnegative soft real numbers class with $\mathbb{R}(P)$ and $\mathbb{R}(P)^*$, respectively.*

**Definition 2.3** *Assume that $(G,P)$ and $(H,P)$ are soft sets on $U$. We call $(G,P)$ a null soft set if $G(\alpha)=\emptyset$ and call $(G,P)$ an absolute soft set if $G(\alpha)=U$ for each $\alpha\in P$, denote them by $\Phi$ and $\tilde{U}$, respectively.*

*We call $(G,P)$ a soft subset of $(H,P)$ if $G(\alpha)\subset H(\alpha)$ for each $\alpha\in P$ and denote by $(G,P)\tilde{\subset}(H,P)$. Then, we call $(H,P)$ a soft upper set of $(G,P)$ and denote by $(H,P)\tilde{\supset}(G,P)$. $(G,P)=(H,P)$ if and only if $(G,P)\tilde{\subset}(H,P)$ and $(H,P)\tilde{\supset}(G,P)$.*

*The union $(F,P)$ of $(G,P)$ and $(H,P)$, denoted by $(F,P)=(G,P)\tilde{\cup}(H,P)$, is defined as $F(\alpha)=G(\alpha)\cup H(\alpha)$. The intersection $(F,P)$ of $(G,P)$ and $(H,P)$, denoted by $(F,P)=(G,P)\tilde{\cap}(H,P)$, is defined as $F(\alpha)=G(\alpha)\cap H(\alpha)$. The difference $(F,P)$ of $(G,P)$ and $(H,P)$, denoted by $(F,P)=(G,P)\tilde{\setminus}(H,P)$, is defined as $F(\alpha)=G(\alpha)\setminus H(\alpha)$. The complement $(G,P)^c=(G^c,P)$ of $(G,P)$ is defined as a mapping $G^c:P\to P(U)$ given by $G^c(\alpha)=U\setminus G(\alpha)$, $\forall\alpha\in P$.*

*The elementary union $(F,P)=(G,P)\Cup(H,P)$ and elementary intersection $(F,P)=(G,P)\Cap(H,P)$ of $(G,P),(H,P)\in S(\tilde{U})$ are defined by $(F,P)=SS(SE(G,P)\cup SE(H,P))$ and $(F,P)=SS(SE(G,P)\cap$*

*$SE(H,P)$), respectively. (For details, see [16, 20, 28, 42]).*

Throughout the work, $\Phi$ and the soft sets $(G,P)$ on $U$ such that $G(\alpha) \neq \emptyset$ for every $\alpha \in P$ will be considered. The class of those soft sets denoted by $S(\tilde{U})$.

### 3. Soft cone metric space

In this section, we briefly introduce soft cone metric space and prove new results for later sections.

**Definition 3.1** *[41] Assume that $(\tilde{U}, \|.\|, P)$ is a soft real Banach space. A soft set $(C,P) \in S(\tilde{U})$ that meets the following conditions is called a soft cone:*

1. *$(C,P) \neq \Phi$, $(C,P) \neq SS(\{\Theta\})$ and $(C,P)$ is closed,*
2. *If $\tilde{x}, \tilde{y} \tilde{\in} (C,P)$ and $\tilde{a}, \tilde{b} \tilde{\in} \mathbb{R}(P)^*$, $\tilde{a}\tilde{x} + \tilde{b}\tilde{y} \tilde{\in} (C,P)$,*
3. *If $\tilde{x} \tilde{\in} (C,P)$ and $-\tilde{x} \tilde{\in} (C,P)$, $\tilde{x} = \Theta$.*

*For a soft cone $(C,P) \in S(\tilde{U})$, a soft partial ordering $\tilde{\preceq}$ according to $(C,P)$ is defined by $\tilde{x} \tilde{\preceq} \tilde{y} \Leftrightarrow \tilde{y} - \tilde{x} \tilde{\in} (C,P)$. Then, we say $\tilde{x} \tilde{\prec} \tilde{y}$ if $\tilde{y} - \tilde{x} \tilde{\in} (C,P)^o = SS(int(C,P))$.*

*We call a soft cone $(C,P)$ in soft real Banach space $\tilde{U}$ normal if for each $\tilde{x}, \tilde{y} \tilde{\in} \tilde{U}$, $\Theta \tilde{\preceq} \tilde{x} \tilde{\preceq} \tilde{y} \Rightarrow \|\tilde{x}\| \tilde{\leq} \tilde{t} \|\tilde{y}\|$, where $\tilde{t} \tilde{>} \bar{0}$ is a soft real number (we say $\tilde{t}$ the soft constant element of $(C,P)$); minihedral if for each $\tilde{x}, \tilde{y} \tilde{\in} \tilde{U}$, $sub(\tilde{x}, \tilde{y})$ exists; strongly minihedral if there is a supremum of each above bounded soft set on $U$; solid if $(C,P)^o$ is not null soft set; regular if each increasing sequence of soft elements of $\tilde{U}$ converges to a soft element in $\tilde{U}$.*

**Example 3.2** *[41] Suppose that $P$ is a finite parameters set and $\mathbb{R}(P)$ (recall that $\mathbb{R}(P) = SE(\tilde{\mathbb{R}})$) is the soft real numbers class. So, $\tilde{\mathbb{R}}^n$ is a soft Banach space. Thus, the soft cone*

$$(C,P) = SS\left\{(\tilde{r}_1, ..., \tilde{r}_n) \in \tilde{\mathbb{R}}^n(P) : \tilde{r}_i \tilde{\geq} \bar{0}\right\}$$

*is normal, solid, minihedral and strongly minihedral.*

**Definition 3.3** *[41] Assume that $P$ is a parameters set and $X$ is a universal set. We call a mapping $d : SE(\tilde{X}) \times SE(\tilde{X}) \to SE(\tilde{U})$ that meets the following conditions, a soft cone metric on $\tilde{X}$ and call $(\tilde{X}, d, P)$ a soft cone metric space. For each $\tilde{x}, \tilde{y}, \tilde{z} \in SE(\tilde{X})$,*

- *D1. $\Theta \tilde{\prec} d(\tilde{x}, \tilde{y})$ and $\Theta = d(\tilde{x}, \tilde{y}) \Leftrightarrow \tilde{x} = \tilde{y}$,*
- *D2. $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x})$,*
- *D3. $d(\tilde{x}, \tilde{y}) \tilde{\preceq} d(\tilde{x}, \tilde{z}) + d(\tilde{z}, \tilde{y})$.*

**Example 3.4** *Suppose that $P$ is a finite parameters set, $(C,P) = SS\left\{(\tilde{x}, \tilde{y}) \tilde{\in} \tilde{\mathbb{R}}^2 : \tilde{x}, \tilde{y} \tilde{\geq} \bar{0}\right\}$ and $d : \mathbb{R}(P) \times \mathbb{R}(P) \to \mathbb{R}^2(P)$ is a mapping defined as $d(\tilde{x}, \tilde{y}) = \left(|\tilde{x} - \tilde{y}|, \tilde{t}\,|\tilde{x} - \tilde{y}|\right)$ for a soft constant $\tilde{t} \tilde{\geq} \bar{0}$ and $\tilde{x}, \tilde{y} \in \mathbb{R}(P)$. Then $(\tilde{\mathbb{R}}, d, P)$ is a soft cone metric space.*

**Theorem 3.5** *[41] Each parametrized class of cone metrics $\{d_\alpha : \alpha \in P\}$ on $X$ gives a soft cone metric on $\tilde{X}$.*

As a result of this theorem, we say that each cone metric on $X$ may be expanded to a soft cone metric on $\tilde{X}$.

**Definition 3.6** *We call $d$ defined by a cone metric $d^*$ a soft cone metric produced by $d^*$.*

**Remark 3.7** *The opposite of Theorem 3.5 is not provided in general. Hence, a soft cone metric is more exhaustive than a parametrized class of cone metrics (see [42]).*

**Theorem 3.8** *[41] Suppose that $d$ is a soft cone metric that meets following condition (D4) on soft set $\tilde{X}$. If for $\alpha \in P$ and $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$, $d_\alpha : X \times X \to U$ is given as $d(\tilde{x}, \tilde{y})(\alpha) = d_\alpha(\tilde{x}(\alpha), \tilde{y}(\alpha))$, $d_\alpha$ is a cone metric.*

*D4. $\{d(\tilde{x}, \tilde{y})(\alpha) : \tilde{x}(\alpha) = a, \tilde{y}(\alpha) = b\}$ is a singleton set if $\alpha \in P$ and $(a, b) \in X \times X$.*

**Theorem 3.9** *Assume that $\left(\tilde{X}, d, P\right)$ is a soft cone metric space, $(C, P) \in S(\tilde{U})$ is a soft set and $(C, P)^\circ = SS(int(C, P))$. There is a soft real number $\tilde{\delta} \tilde{>} \bar{0}$ satisfying $\tilde{c} - \tilde{x} \tilde{\in} (C, P)^o$ for each $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ in $\left(\tilde{X}, d, P\right)$, where $\|\tilde{x}\| \tilde{<} \tilde{\delta}$.*

**Proof** Since $\Theta \tilde{\prec} \tilde{c}$, $\tilde{c} \in int(C, P)$. Hence, there is a $\tilde{\delta} \tilde{>} \bar{0}$ satisfying $\left\{\tilde{x} \tilde{\in} \tilde{U} : \|\tilde{x} - \tilde{c}\| \tilde{<} \tilde{\delta}\right\}$ in $int(C, P)$. When $\|\tilde{x}\| \tilde{<} \tilde{\delta}$, $\|(\tilde{c} - \tilde{x}) - \tilde{c}\| = \|\tilde{x}\| \tilde{<} \tilde{\delta}$. So, $\tilde{c} - \tilde{x} \in int(C, P)$. Thus $\tilde{c} - \tilde{x} \tilde{\in} (C, P)^o$. □

**Theorem 3.10** *There is $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ satisfying $\tilde{c} \tilde{\prec} \tilde{c}_1$ and $\tilde{c} \tilde{\prec} \tilde{c}_2$ for $\Theta \tilde{\prec} \tilde{c}_1, \tilde{c}_2 \tilde{\in} \tilde{U}$ in $\left(\tilde{X}, d, P\right)$.*

**Proof** Since $\Theta \tilde{\prec} \tilde{c}_2$, from Theorem 3.9 there is any $\tilde{\delta} \tilde{>} \bar{0}$ satisfying $\|\tilde{x}\| \tilde{<} \tilde{\delta}$. Hence, $\tilde{x} \tilde{\prec} \tilde{c}$. If $n_0 \in \mathbb{N}$ is selected with $\frac{\bar{1}}{n_0} \tilde{<} \frac{\tilde{\delta}}{\|\tilde{c}_1\|}$. Let $\tilde{c} = \frac{\tilde{c}_1}{n_0}$. Then $\|\tilde{c}\| = \left\|\frac{\tilde{c}_1}{n_0}\right\| = \frac{\|\tilde{c}_1\|}{n_0} \tilde{<} \tilde{\delta}$. From here, we get $\tilde{c} \tilde{\prec} \tilde{c}_2$. Obviously, $\tilde{c} \tilde{\prec} \tilde{c}_1$ and $\Theta \tilde{\prec} \tilde{c}$. □

## 4. Topology of soft cone metric space

**Definition 4.1** *[4] Assume that $(\tilde{X}, d, P)$ is a soft cone metric space $\tilde{x} \tilde{\in} \tilde{X}$ and $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$. We call the class of soft elements*

$$B(\tilde{x}, \tilde{c}) = \left\{\tilde{z} \tilde{\in} \tilde{X} : d(\tilde{z}, \tilde{x}) \tilde{\prec} \tilde{c}\right\} \subset SE(\tilde{X}).$$

*and $(B_{\tilde{c}}, P) = SS(B(\tilde{x}, \tilde{c}))$ an open ball and a soft open ball, respectively.*

*Similarly, we call the class of soft elements*

$$B[\tilde{x}, \tilde{c}] = \left\{\tilde{z} \tilde{\in} \tilde{X} : d(\tilde{z}, \tilde{x}) \tilde{\preceq} \tilde{c}\right\} \subset SE(\tilde{X}).$$

*and $[B_{\tilde{c}}, P] = SS(B(\tilde{x}, \tilde{c}))$ a closed ball and a soft closed ball, respectively. From this definition, $\tilde{c} - d(\tilde{x}, \tilde{z}) \tilde{\in} (G, P)^\circ$ for each $\tilde{z} \in B(\tilde{x}, \tilde{c})$ and $\tilde{c} - d(\tilde{x}, \tilde{z}) \tilde{\in} (G, P)$ for each $\tilde{z} \tilde{\in} (B_{\tilde{c}}, P)$.*

**Definition 4.2** *Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space,* $\Upsilon \subset SE(\tilde{X})$ *is a class of soft elements. We call* $\tilde{x} \in \Upsilon$ *an interior element of* $\Upsilon$ *if there is* $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ *satisfying* $\tilde{x} \in B(\tilde{x}, \tilde{c}) \subset \Upsilon$.

**Definition 4.3** *Assume that* $(G, P) \in S(\tilde{X})$ *is a soft subset in* $(\tilde{X}, d, P)$. *We call* $\tilde{x} \tilde{\in} (G, P)$ *an interior element of* $(G, P)$ *if there is* $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ *satisfying* $\tilde{x} \in B(\tilde{x}, \tilde{c}) \subset SE(G, P)$.

*The interior* $int(G, P)$ *of* $(G, P)$ *is defined as the class containing of all interior elements of* $(G, P)$. $(G, P)^\circ = SS\left(int(G, P)\right)$ *is called the soft interior.*

**Theorem 4.4** *Suppose that* $(G, P), (H, P) \in S(\tilde{X})$ *is soft subsets in* $(\tilde{X}, d, P)$. *The following properties are provided.*

1. $(G, P)^\circ \tilde{\subset} (G, P)$,
2. $(G, P) \tilde{\subset} (H, P)$ *implies* $int(G, P) \subset int(H, P)$,
3. $int(G, P) \cap int(H, P) \subset int\left((G, P) \Cap (H, P)\right)$,
4. $int(G, P) \cup int(H, P) \subset int\left((G, P) \Cup (H, P)\right)$.

**Proof** Only the property 3 will be proved.

Let $\tilde{x} \in int(G, P) \cap int(H, P)$. Then, $\tilde{x} \in int(G, P)$ and $\tilde{x} \in int(H, P)$. Hence, there are $\Theta \tilde{\prec} \tilde{c}_1, \tilde{c}_2 \tilde{\in} \tilde{U}$ satisfying $\tilde{x} \in B(\tilde{x}, \tilde{c}_1) \subset SE(G, P)$ and $\tilde{x} \in B(\tilde{x}, \tilde{c}_2) \subset SE(H, P)$. Put $\tilde{c}(\alpha) = min\{\tilde{c}_1(\alpha), \tilde{c}_2(\alpha)\}$ for each $\alpha \in P$. Then, $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ and $\tilde{x} \in B(\tilde{x}, \tilde{c}) \subset SE(G, P)$ and $\tilde{x} \in B(\tilde{x}, \tilde{c}) \subset SE(H, P)$ implies $\tilde{x} \in int((G, P) \Cap (H, P))$. Thus, $int(G, P) \cap int(H, P) \subset int((G, P) \Cap (H, P))$. □

**Definition 4.5** *We call* $\Upsilon \subset SE(\tilde{X})$ *an open class in* $(\tilde{X}, d, P)$ *if each element of* $\Upsilon$ *is interior element of itself.*

*Suppose that* $(G, P) \in S(\tilde{X})$. *We call* $(G, P)$ *a soft open set in* $(\tilde{X}, d, P)$ *if there is an open class* $\Upsilon$ *of soft elements of* $(G, P)$ *and* $(G, P) = SS(\Upsilon)$.

**Example 4.6** *In* $(\tilde{X}, d, P)$, *an open ball* $B(\tilde{x}, \tilde{c})$ *is an open set so,* $(B_{\tilde{c}}, P)$ *is soft open set.*

**Theorem 4.7** *[4] The followings are provided in* $(\tilde{X}, d, P)$,

1. $\tilde{X}$ *and* $\Phi$ *are soft open.*
2. *Arbitrary elementary union of soft open sets is soft open.*

The finite elementary intersection of soft open sets can not be soft open [20]. However, we have the following propositions, if $d$ meets the condition $(D4)$.

**Proposition 4.8** *[4] In* $(\tilde{X}, d, P)$ *that meets the condition* $(D4)$, $(B_{\tilde{c}}, P)(\alpha) = B(\tilde{x}(\alpha), \tilde{c}(\alpha))$ *is an open ball in* $(\tilde{X}, d_\alpha)$ *for all* $\alpha \in P$, $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ *and open ball* $B(\tilde{x}, \tilde{c})$.

**Proposition 4.9** *[4] In $(\tilde{X}, d, P)$ that meets the condition $(D4)$, $(G,P) \in S(\tilde{X})$ is soft open set if and only if $(G,P)(\alpha)$ is open in $(\tilde{X}, d_\alpha)$ for each $\alpha \in P$.*

**Theorem 4.10** *[4] In $(\tilde{X}, d, P)$ that meets the condition $(D4)$, the finite elementary intersection of soft open sets is soft open.*

**Remark 4.11** *The arbitrary elementary intersection of soft open sets can not be soft open. For example, see Example 3.4. Consider the soft sets $(G,P)_n$ with $(G,P)_n(\alpha) = (0, \frac{1}{n})$ for each $\alpha \in P$ and $n \in \mathbb{N}$. Then $(G,P) = \Cap_{n\in\mathbb{N}}(G,P)_n = \{\bar{0}\}$. Since there is not a soft element $\Theta \tilde{\precsim} \tilde{c} \tilde{\in} \tilde{U}$ satisfying $\bar{0} \in B(\bar{0}, \tilde{c}) \subset SE(\{\bar{0}\})$, $(G,P)$ is not soft open.*

**Definition 4.12** *[42] Assume that $\tilde{X}$ is an absolute soft set. We call a soft sets class $\tau \subset S(\tilde{X})$ an elementary soft topology on $\tilde{X}$ and call the triple $\left(\tilde{X}, \tau, P\right)$ an elementary soft topological space if the following axioms are satisfied.*

*i.* $\tilde{X}, \Phi \in \tau$,

*ii.* $\{(U_i, P)\}_{i\in I} \subset \tau$ *implies* $\Cup_{i\in I}(U_i, P) \in \tau$,

*iii.* $\{(U_i, P)\}_{i=1}^{n} \subset \tau$ *implies* $\Cap_{i=1}^{n}(U_i, P) \in \tau$.

**Theorem 4.13** *Any soft cone metric space that meets the condition $(D4)$ is an elementary soft topological space.*

**Proof** Assume that $\tau_{\tilde{c}} \subset S(\tilde{X})$ is the class of entire soft open sets in $(\tilde{X}, d, P)$ that meets the condition $(D4)$. From Theorems 4.7 and 4.10, $\tau_{\tilde{c}}$ is a soft topology on $\tilde{X}$ in accordance with elementary intersection and elementary union of soft sets. □

**Definition 4.14** *We call the topology $\tau_{\tilde{c}}$ the elementary soft metric topology on $\tilde{X}$ and call $(\tilde{X}, \tau_{\tilde{c}}, P)$ the elementary soft metric topological space.*

**Definition 4.15** *We call $(G,P) \in S(\tilde{X})$ a soft closed set in $(\tilde{X}, d, P)$ if $(G,P)^C \in S(\tilde{X})$ and $(G,P)^{\mathbb{C}} \in \tau_{\tilde{c}}$.*

**Example 4.16** *We consider Example 3.4. Let $(G,P) = \{\tilde{x}\}$. Then $(G,P)^C \in S(\tilde{X})$ and $(G,P)^{\mathbb{C}}$ is open in $(\tilde{X}, d, P)$, where $(G,P)^C(\alpha) = (-\infty, \tilde{x}) \cup (\tilde{x}, \infty)$. Because for all $\tilde{y} \tilde{\in} (G,P)^C$, there is a soft element $\Theta \tilde{\precsim} \tilde{c} \tilde{\in} \tilde{U}$ satisfying $\tilde{y} \in B(\tilde{y}, \tilde{c}) \subset SE((G,P)^C)$. So $\tilde{y} \in int\left((G,P)^C\right)$ implies $\tilde{y} \tilde{\in} \left((G,P)^C\right)^o$. Hence $(G,P)^C \tilde{\subset} \left((G,P)^C\right)^o$. But $\left((G,P)^C\right)^o \tilde{\subset} (G,P)^C$ from Theorem 4.4. Thus $(G,P)^C = \left((G,P)^C\right)^o$ i.e. $(G,P)^C$ is open. $(G,P)^{\mathbb{C}} = (G,P)^C$ since $(G,P)^{\mathbb{C}} \neq \Phi$, so $(G,P)^{\mathbb{C}}$ is soft open. Thus $(G,P)$ is soft closed.*

**Theorem 4.17** *The followings are provided in $(\tilde{X}, d, P)$.*

1. $\tilde{X}$ and $\Phi$ are soft closed.

2. Arbitrary elementary intersection of soft closed sets is soft closed.

**Proof**

1. $\tilde{X}$ is soft closed, since $\tilde{X}^{\mathbb{C}} = \Phi$ is soft open and $\tilde{X}^{C} \in S(\tilde{X})$. Similarly $\Phi$ is soft closed.

2. If for $i \in I$, $(G_i, P) \in S(\tilde{X})$ are soft closed, $(G_i, P)^C \in S(\tilde{X})$ and $(G_i, P)^{\mathbb{C}}$ is soft open. We must show that $(G, P) = \Cap_{i\in I}(G_i, P)$ is soft closed. For $I = \varnothing$, we get $(G, P) = \Cap_{i\in I}(G_i, P) = \tilde{X}$, which is soft closed. For $I \neq \varnothing$ and each $i \in I$, $(G_i, P) = \Phi$ then $(G, P) = \Cap_{i\in I}(G_i, P) = \Phi$ is soft closed. Lastly, $I \neq \varnothing$ and $(G, P) \neq \Phi$, then $(G, P) = \Cap_{i\in I}(G_i, P) = \tilde{\cap}_{i\in I}(G_i, P)$. Hence,

$$(G, P)^C = \left(\tilde{\cap}_{i\in I}(G_i, P)\right)^C = \tilde{\cup}_{i\in I}(G_i, P)^C = \Cup_{i\in I}(G_i, P)^C$$

is soft open by Theorem 4.7 and is a nonnull member of $S(\tilde{X})$. Since $(G, P)^{\mathbb{C}} = SS\left(SE(G, P)^C\right)$ is soft open, $(G, P)$ is soft closed.

□

**Theorem 4.18** *In $(\tilde{X}, d, P)$ that meets the condition $(D4)$, the finite elementary union of soft closed sets is soft closed.*

**Proof** It is sufficient to prove the theorem for two closed sets. Since $(\tilde{X}, d, P)$ that meets the condition $(D4)$ and $(G, P), (H, P) \in S(\tilde{X})$ are two soft closed sets, then $(G, P)^C, (H, P)^C \in S(\tilde{X})$ and $(G, P)^{\mathbb{C}}$ and $(H, P)^{\mathbb{C}}$ are soft open sets. So $(G, P)^{\mathbb{C}} \Cap (H, P)^{\mathbb{C}} = ((G, P) \Cup (H, P))^{\mathbb{C}}$ is soft open by Theorem 4.10 and hence $(G, P) \Cup (H, P)$ is soft closed set in $(\tilde{X}, d, P)$. □

**Definition 4.19** *Assume that $(G, P) \in S(\tilde{X})$ is a soft set in $(\tilde{X}, d, P)$, $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ and $\tilde{x} \tilde{\in} \tilde{X}$. We call $\tilde{x}$ a soft closure element of $(G, P)$ if $B(\tilde{x}, \tilde{c}) \cap SE(G, P) \neq \varnothing$ for any soft open ball $B(\tilde{x}, \tilde{c})$ containing $\tilde{x}$.*

*We call that $\tilde{x}$ is not a soft closure element of $(G, P)$ if there is a soft open ball $B(\tilde{x}, \tilde{c})$ containing $\tilde{x}$ with $B(\tilde{x}, \tilde{c}) \cap SE(G, P) = \varnothing$ and $(B_{\tilde{c}}, P) \Cap (G, P) \in S(\tilde{X})$.*

*The class of all soft closure element of $(G, P)$ is denoted by $cl(G, P)$. $\overline{(G, P)} = SS\,(cl(G, P))$ is called as to soft closure of $(G, P)$.*

**Theorem 4.20** *Suppose that $(G, P), (H, P) \in S(\tilde{X})$ are soft sets in $(\tilde{X}, d, P)$. Then the followings are valid.*

1. $\overline{(G, P)} = \Cap \left\{(H, P) \in S(\tilde{X}) : (G, P) \tilde{\subset} (H, P),\ (H, P) \text{ is soft closed}\right\}$,

2. $(G, P)$ is closed soft set iff $(G, P) = \overline{(G, P)}$,

3. $\bar{\Phi} = \Phi$ and $\overline{\tilde{X}} = \tilde{X}$,

4. $(G, P) \tilde{\subset} \overline{(G, P)}$,

5. $\overline{\overline{(G, P)}} = (G, P)$,

6. $(G,P)\tilde{\subset}(H,P)$ implies $cl(G,P)\subset cl(H,P)$,

7. $\overline{(G,P)}^{\mathbb{C}} \Cap \overline{(H,P)}^{\mathbb{C}} \neq \Phi$ implies $\overline{(G,P)} \Cup \overline{(H,P)} = \overline{(G,P)\Cup(H,P)}$,

8. $\overline{(G,P)} \Cap \overline{(H,P)} \ \tilde{\subset}\ \overline{(G,P)\Cap(H,P)}$.

**Proof** Only the property 7 will be proved. Since $(G,P)\tilde{\subset}(G,P)\Cup(H,P)$ and $(H,P)\tilde{\subset}(G,P)\Cup(H,P)$,

$$\overline{(G,P)}\tilde{\subset}\overline{(G,P)\Cup(H,P)} \quad \text{and} \quad \overline{(H,P)}\tilde{\subset}\overline{(G,P)\Cup(H,P)}.$$

Hence $\overline{(G,P)}\Cup\overline{(H,P)}\tilde{\subset}\overline{(G,P)\Cup(H,P)}$.

On the other hand, since $(G,P)\tilde{\subset}\overline{(G,P)}$ and $(H,P)\tilde{\subset}\overline{(H,P)}$ from property 4, $(G,P)\Cup(H,P)\tilde{\subset}\overline{(G,P)}\Cup\overline{(H,P)}$. If $\overline{(G,P)}^{\mathbb{C}} \Cap \overline{(H,P)}^{\mathbb{C}} \neq \Phi$, by Theorem 4.18 $\overline{(G,P)}\Cup\overline{(H,P)}$ is closed soft set. By property 1, $\overline{(G,P)\Cup(H,P)}\tilde{\subset}\overline{(G,P)}\Cup\overline{(H,P)}$ since $\overline{(G,P)\Cup(H,P)}$ is the smallest soft set containing the $(G,P)\Cup(H,P)$. □

## 5. Compactness of soft cone metric space

**Definition 5.1** *[41] Assume that $\{\tilde{x}_n\}$ is a sequence of soft elements in $(\tilde{X},d,P)$ and $\tilde{x}\tilde{\in}\tilde{X}$.*

1. *We call $\{\tilde{x}_n\}$ convergent to $\tilde{x}$ if there is a $N\in\mathbb{N}$ satisfying $d(\tilde{x}_n,\tilde{x})\tilde{\prec}\tilde{c}$ for any $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$ and all $n>N$, and denote by $\tilde{x}_n\to\tilde{x}$ as $n\to\infty$.*

2. *We call $\{\tilde{x}_n\}$ a Cauchy sequence in $\tilde{X}$ if there is a $N\in\mathbb{N}$ satisfying $d(\tilde{x}_n,\tilde{x}_m)\tilde{\prec}\tilde{c}$ for any $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$ and all $n,m>N$ .*

3. *We call $(\tilde{X},d,P)$ complete if any Cauchy sequence $\{\tilde{x}_n\}$ converges a soft element in $\tilde{X}$.*

**Proposition 5.2** *[41] Assume that $(C,P)$ is a normal soft cone with soft constant $\tilde{t}$ in $(\tilde{X},d,P)$ and $\tilde{x}\tilde{\in}\tilde{X}$. Then,*

1. *A sequence $\{\tilde{x}_n\}$ of soft elements of $\tilde{X}$ converges to $\tilde{x}$ if and only if $d(\tilde{x}_n,\tilde{x})\to\Theta$ as $n\to\infty$.*

2. *Let us consider two sequences $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ of soft elements of $\tilde{X}$ with $\tilde{x}_n\to\tilde{x}$ and $\tilde{y}_n\to\tilde{y}$ as $n\to\infty$. Then $d(\tilde{x}_n,\tilde{y}_n)\to(\tilde{x},\tilde{y})$ as $n\to\infty$.*

**Definition 5.3** *Assume that $(\tilde{X},d,P)$ is a soft cone metric space. We call a soft set $(G,P)\in S(\tilde{X})$ sequentially closed if any sequence of soft elements of $(G,P)$ converges to a soft element in $(G,P)$.*

**Proposition 5.4** *In $(\tilde{X},d,P)$, the closed ball $B[\tilde{x},\tilde{c}]$ and soft closed ball $[B_{\tilde{c}},P]$ are sequentially closed for $\tilde{x}\tilde{\in}\tilde{X}$ and $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$.*

**Proof** If $\{\tilde{y}_n\}$ is a sequence of soft elements in $B[\tilde{x},\tilde{c}]$ with $\tilde{y}_n\to\tilde{y}$ as $n\to\infty$, $d(\tilde{y}_n,\tilde{x})\to\Theta$ and $d(\tilde{y}_n,\tilde{x})\tilde{\preceq}\tilde{c}$ as $n\to\infty$. Hence

$$\tilde{y}\in B[\tilde{x},\tilde{c}] \Leftrightarrow d(\tilde{y},\tilde{x})\tilde{\preceq}\tilde{c} \Leftrightarrow \tilde{c}-d(\tilde{y},\tilde{x})\tilde{\in}(C,P).$$

But, we obtain $d(\tilde{y}_n,\tilde{x})\to d(\tilde{y},\tilde{x})$ from Proposition 5.2. So, $lim_{n\to\infty}(\tilde{c}-d(\tilde{y}_n,\tilde{x}))=\tilde{c}-d(\tilde{y},\tilde{x})\tilde{\in}(C,P)$ since $(C,P)$ is soft closed. Thus $B[\tilde{x},\tilde{c}]$ is sequentially closed and so $[B_{\tilde{c}},P]$ is sequentially soft closed. □

**Definition 5.5** *Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space. We call a soft set* $(G, P) \in S(\tilde{X})$ *bounded above if there is* $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ *satisfying* $d(\tilde{x}, \tilde{y}) \tilde{\preceq} \tilde{c}$ *for all* $\tilde{x}, \tilde{y} \tilde{\in} (G, P)$ *and we call it bounded if*

$$\delta(G, P) = \sup \{d(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \tilde{\in} (G, P)\} \tilde{\in} \tilde{U}.$$

*We call* $(G, P)$ *unbounded soft set if the supremum does not exist in* $\tilde{U}$.

**Theorem 5.6** *Suppose that* $(C, P)$ *is a strongly minihedral cone with soft constant* $\tilde{t}$ *in* $(\tilde{X}, d, P)$. *A soft set* $(G, P) \in S(\tilde{X})$ *is bounded if and only if*

$$\delta(G, P)(\alpha) = \sup_{\tilde{x}, \tilde{y} \tilde{\in} (G,P)} \|d_\alpha(\tilde{x}(\alpha), \tilde{y}(\alpha))\| < \infty, \forall \alpha \in P.$$

**Proof** If $(G, P) \in S(\tilde{X})$ is a bounded soft set, there is $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ satisfying $d(\tilde{x}, \tilde{y}) \tilde{\preceq} \tilde{c}$ for every $\tilde{x}, \tilde{y} \tilde{\in} (G, P)$. Then for every $\alpha \in P$

$$\begin{aligned} \|d(\tilde{x}, \tilde{y})\| \tilde{\leq} \tilde{t} \|\tilde{c}\| \quad &\Rightarrow \quad \|d(\tilde{x}, \tilde{y})\| (\alpha) \leq \tilde{t} \|\tilde{c}\| (\alpha), \\ &\Rightarrow \quad \|d_\alpha(\tilde{x}(\alpha), \tilde{y}(\alpha))\| \leq \tilde{t}(\alpha) \|\tilde{c}(\alpha)\|, \\ &\Rightarrow \quad \|d_\alpha(\tilde{x}(\alpha), \tilde{y}(\alpha))\| < \infty. \end{aligned}$$

So, $\delta(G, P)(\alpha) = \sup_{\tilde{x}, \tilde{y} \tilde{\in} (G,P)} \|d_\alpha(\tilde{x}(\alpha), \tilde{y}(\alpha))\| < \infty$.

Conversely, let us assume that for each $\alpha \in P$

$$\delta(G, P)(\alpha) = \sup_{\tilde{x}, \tilde{y} \tilde{\in} (G,P)} \|d_\alpha(\tilde{x}(\alpha), \tilde{y}(\alpha))\| = M_\alpha < \infty.$$

Let us get fixed some $k_\alpha \in U$ with $\theta \prec k_\alpha$. Then there exists $\tilde{k} \tilde{\in} \tilde{U}$ with $\Theta \tilde{\prec} \tilde{k}$ such that for any $\alpha \in P$, $\tilde{k}(\alpha) = k_\alpha$. Hence, by Theorem 3.10, $\tilde{\delta} \tilde{>} \bar{0}$ can be found so that $\|\tilde{z}\| \tilde{<} \tilde{\delta}$. Thus, $\tilde{z} \tilde{\in} \tilde{U}$ with $\tilde{z} \tilde{\prec} \tilde{k}$. For each $\tilde{x}, \tilde{y} \tilde{\in} (G, P)$ let $\tilde{c} = \frac{\tilde{\delta} d(\tilde{x}, \tilde{y})}{2\|d(\tilde{x}, \tilde{y})\|}$. Then $\|\tilde{c}\| = \frac{\tilde{\delta}}{2} < \tilde{\delta}$. Hence $\frac{\tilde{\delta} d(\tilde{x}, \tilde{y})}{2\|d(\tilde{x}, \tilde{y})\|} \tilde{\prec} \tilde{k}$ and so $\tilde{k} - \frac{\tilde{\delta} d(\tilde{x}, \tilde{y})}{2\|d(\tilde{x}, \tilde{y})\|} \tilde{\in} (C, P)^o$. Therefore $\frac{2\|d(\tilde{x}, \tilde{y})\|}{\tilde{\delta}} \tilde{k} - \frac{2\|d(\tilde{x}, \tilde{y})\|}{\tilde{\delta}} \frac{\tilde{\delta} d(\tilde{x}, \tilde{y})}{2\|d(\tilde{x}, \tilde{y})\|} \tilde{\in} (C, P)^o$, then $\frac{2\|d(\tilde{x}, \tilde{y})\|}{\tilde{\delta}} \tilde{k} - d(\tilde{x}, \tilde{y}) \tilde{\in} (C, P)^o$ i.e. $d(\tilde{x}, \tilde{y}) \tilde{\prec} \frac{2\|d(\tilde{x}, \tilde{y})\|}{\tilde{\delta}} \tilde{k} \tilde{\leq} \frac{2\tilde{M}}{\tilde{\delta}} \tilde{k} = \tilde{c} \tilde{\in} (C, P)^o$, where $\tilde{M}(\alpha) = M_\alpha, \forall \alpha \in P$ and so $d(\tilde{x}, \tilde{y}) \tilde{\preceq} \tilde{c}$. Thus, $(G, P)$ is bounded, since $(C, P)$ is strongly minihedral. □

**Definition 5.7** *Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space,* $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ *and* $\Upsilon = \{\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n\}$ *is a finite class of soft elements of* $\tilde{X}$ *and* $(U, P) = SS(\Upsilon)$ *We call* $(U, P)$ *a soft* $\tilde{c} - net$ *for a soft set* $(G, P) \in S(\tilde{X})$ *if each* $\tilde{x} \tilde{\in} (G, P)$ *there is* $\tilde{u}_{i_o} \tilde{\in} (U, P)$ *such that* $d(\tilde{x}, \tilde{u}_{i_o}) \tilde{\prec} \tilde{c}$.

**Definition 5.8** *Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space and* $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$. *We call a soft set* $(G, P) \in S(\tilde{X})$ *totally bounded if* $(G, P) \tilde{\subset} \Cup_{i=1}^{n} (U_i, P)$, *where* $\delta(U_i, P) \tilde{\prec} \tilde{c}$.

**Theorem 5.9** *Suppose that* $(G, P) \in S(\tilde{X})$ *is a soft set in* $(\tilde{X}, d, P)$. *Then* $(G, P)$ *totally bounded if and only if* $(G, P)$ *has a soft* $\tilde{c}$*-net for each* $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$.

**Proof** Let us assert that $(G,P)$ totally bounded and $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$. So, we can find $(U_i,P)$ satisfying $(G,P)\tilde{\subset}\Cup_{i=1}^{n}$ $(U_i,P)$, where $\delta(U_i,P)\tilde{\prec}\tilde{c}$. Let us create a class $\Upsilon$ by selecting a soft element $\tilde{u}_i$ from each set $(U_i,P)$. Take $(U,P)=SS(\Upsilon)$. We prove that $(U,P)$ is a soft $\tilde{c}-net$ for $(G,P)$. Let $\tilde{x}\tilde{\in}(G,P)$. There is $\tilde{u}_{i_o}$, $i_o\in\{1,2,...,n\}$, such that $\tilde{x}\tilde{\in}(U_{i_o},P)$. We finalize that $d(\tilde{x},\tilde{u}_{i_o})\tilde{\prec}\tilde{c}$, since $\tilde{x},\tilde{u}_{i_o}\tilde{\in}(U_{i_o},P)$ and $\delta(U_i,P)\tilde{\prec}\tilde{c}$.

Conversely, give $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$. Then we can find a finite class $\Upsilon=\{\tilde{u}_1,\tilde{u}_2,...,\tilde{u}_n\}$ of soft elements of $\tilde{X}$ and a soft set $(U,P)=SS(\Upsilon)$ such that there is $\tilde{u}_{i_o}\tilde{\in}(U,P)$ with $d(\tilde{x},\tilde{u}_{i_o})\tilde{\prec}\tilde{c}$ for each $\tilde{x}\tilde{\in}(G,P)$. Let $(U_{i_o},P)=(B_{\tilde{c}},P)=SS\left\{\tilde{y}\tilde{\in}\tilde{X}:d(\tilde{y},\tilde{u}_{i_o})\tilde{\prec}\tilde{c}\right\}$. Clearly $(G,P)\tilde{\subset}\Cup_{i=1}^{n}(U_i,P)$ and $\delta(U_i,P)\tilde{\prec}\tilde{c}$. □

**Definition 5.10** *Assume that $(\tilde{X},d,P)$ is a soft cone metric space. We call a soft set $(G,P)\in S(\tilde{X})$ sequentially compact if there is a subsequence $\{\tilde{x}_{n_i}\}$ of any sequence $\{\tilde{x}_n\}$ of soft element in $(G,P)$ such that $\{\tilde{x}_{n_i}\}$ converges to a soft element in $(G,P)$.*

**Proposition 5.11** *Any sequentially soft closed set $(G,P)\in S(\tilde{X})$ in a sequentially compact soft cone metric space $(\tilde{X},d,P)$ is sequentially compact.*

**Proof** Assume that $\{\tilde{x}_n\}$ is a sequence of soft elements of $(G,P)$, it is also a sequence of soft elements of $\tilde{X}$. Hence, it has a subsequence converging to a soft element of $\tilde{X}$ since $\tilde{X}$ is sequentially compact. Thus, $\tilde{x}\tilde{\in}(G,P)$ since $(G,P)$ is sequentially soft closed. □

**Proposition 5.12** *In $(\tilde{X},d,P)$, a soft set $(G,P)\in S(\tilde{X})$ is totally bounded if $(G,P)$ is sequentially compact.*

**Proof** We suppose that $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$, so $(G,P)$ has no soft $\tilde{c}$-net. If $\tilde{x}_1\tilde{\in}(G,P)$ is a fixed soft element, there is $\tilde{x}_2\tilde{\in}(G,P)$ with $\tilde{c}-d(\tilde{x}_1,\tilde{x}_2)\tilde{\notin}(C,P)^o$ and $SS\{\tilde{x}_1,\tilde{x}_2\}$ has no soft $\tilde{c}$-net. Thus, there is $\tilde{x}_3\tilde{\in}(G,P)$ with $\tilde{c}-d(\tilde{x}_1,\tilde{x}_3)\tilde{\notin}(C,P)^o$ and $\tilde{c}-d(\tilde{x}_2,\tilde{x}_3)\tilde{\notin}(C,P)^o$. Continuing in this way, a sequence $\{\tilde{x}_n\}$ of soft element is obtained in $(G,P)$ with $\tilde{c}-d(\tilde{x}_n,\tilde{x}_m)\tilde{\notin}(C,P)^o$ for each $n,m\in\mathbb{N}$, which $\{\tilde{x}_n\}$ has nonconvergent Cauchy subsequence. This contrasts with the sequentially compactness of $(G,P)$. Thus, $(G,P)$ is not sequentially compact. □

**Lemma 5.13** *Suppose that $(C,P)$ is a soft cone in $\tilde{U}$ and $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$ are two sequences in $\tilde{U}$. If $\tilde{x}_n\to\tilde{x}$, $\tilde{y}_n\to\tilde{y}$ as $n\to\infty$ in $\left(\tilde{U},\|.\|,P\right)$ and $\tilde{x}_n\tilde{\preceq}\ \tilde{y}_n$ for each $n\in\mathbb{N}$, then $\tilde{x}\tilde{\preceq}\ \tilde{y}$.*

**Proof** If $\tilde{x}_n\tilde{\preceq}\ \tilde{y}_n$, $(\tilde{y}_n-\tilde{x}_n)\tilde{\in}(C,P)$. So, $(\tilde{y}_n-\tilde{x}_n)\to(\tilde{y}-\tilde{x})$ and hence $(\tilde{y}-\tilde{x})\tilde{\in}(C,P)$ since $(C,P)$ is soft closed cone. Thus $\tilde{x}\tilde{\preceq}\ \tilde{y}$. □

**Proposition 5.14** *If $(G,P)\in S(\tilde{X})$ is a sequentially compact soft set in $(\tilde{X},d,P)$, there are $\tilde{x}_o,\tilde{y}_0\tilde{\in}(G,P)$ with $\delta(G,P)=\sup\{d(\tilde{x},\tilde{y}):\tilde{x},\tilde{y}\tilde{\in}(G,P)\}=d(\tilde{x}_o,\tilde{y}_0)$.*

**Proof** For a fixed soft element $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$ and $n\in\mathbb{N}$ we have $\delta(G,P)-\frac{\tilde{c}}{n}\tilde{\prec}\delta(G,P)$. From definition of supremum, we can find $\tilde{x}_n,\tilde{y}_n\tilde{\in}(G,P)$ such that $\delta(G,P)-\frac{\tilde{c}}{n}\tilde{\prec}d(\tilde{x}_n,\tilde{y}_n)\tilde{\preceq}\delta(G,P)$ for each $n\in\mathbb{N}$. From sequential compactness, there are sequences $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ with $\tilde{x}_n \to \tilde{x}_o$, $\tilde{y}_n \to \tilde{y}_o$ as $n \to \infty$. By Lemma 5.13,

$$\lim_{n\to\infty}\left(\delta(G,P)-\frac{\tilde{c}}{n}\right)\tilde{\prec}\lim_{n\to\infty}\left(d\left(\tilde{x}_n,\tilde{y}_n\right)\right)\tilde{\preceq}\delta(G,P).$$

Hence $\delta(G,P)\tilde{\prec}\lim_{n\to\infty}\left(d\left(\tilde{x}_n,\tilde{y}_n\right)\right)\tilde{\preceq}\delta(G,P)$. So $\delta(G,P)-\lim_{n\to\infty}\left(d\left(\tilde{x}_n,\tilde{y}_n\right)\right)\tilde{\in}(C,P)$. Thus,

$$\lim_{n\to\infty}\left(d\left(\tilde{x}_n,\tilde{y}_n\right)\right)-\delta(G,P)\tilde{\in}(C,P)^o.$$

By the axiom 3 of Definition 3.1, $\delta(G,P)=\sup\left\{d(\tilde{x},\tilde{y}):\tilde{x},\tilde{y}\tilde{\in}(G,P)\right\}=d\left(\tilde{x}_o,\tilde{y}_0\right)$. □

**Definition 5.15** *Assume that* $(\tilde{X},\tau_{\tilde{c}},P)$ *is an elementary soft metric topological space and*

$$\mathscr{L}=\{(V_i,P):i\in I\}\subset\tau_{\tilde{c}}$$

*is a class of soft set. We call* $\mathscr{L}$ *a soft open cover for a soft set* $(G,P)\in S(\tilde{X})$ *if* $(G,P)\tilde{\subset}\Cup_{i\in I}\left(V_i,P\right)$.

**Definition 5.16** *Assume that* $(\tilde{X},\tau_{\tilde{c}},P)$ *is an elementary soft metric topological space and* $(G,P)\in S(\tilde{X})$ *is a soft set. We call a soft element* $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$ *a Lebesgue soft element of a soft open cover* $\mathscr{L}=\{(V_i,P):i\in I\}\subset\tau_{\tilde{c}}$ *for* $(G,P)$ *if for each* $(H,P)\in S(G,P)$ *with* $\delta(H,P)\tilde{\prec}\tilde{c}$ *there exists* $(V_{i_o},P)\in\mathscr{L}$ *such that* $(H,P)\tilde{\subset}\left(V_{i_o},P\right)$.

**Definition 5.17** *Assume that* $(\tilde{X},\tau_{\tilde{c}},P)$ *is an elementary soft metric topological space and* $(G,P)\in S(\tilde{X})$ *is a soft set. We call* $(G,P)$ *compact if there is a finite soft subcover of each soft open cover for* $(G,P)$.

**Remark 5.18** *The above terminology will be used to prove that sequentially compact soft sets on a soft cone metric space that meets the condition* $(D4)$ *are compact. Since all soft cone metric space is not an elementary soft topological space, in general, a sequentially compact soft cone metric space can not be compact. Because, with respect to Definitions 5.15–5.17, compactness depends to the elementary soft topology on the soft metric space.*

**Theorem 5.19** *Suppose that* $(G,P)\in S(\tilde{X})$ *is a sequentially compact soft set in* $(\tilde{X},d,P)$ *that meets the condition* $(D4)$ *and* $\tau_{\tilde{c}}$ *is the elementary soft metric topology on* $\tilde{X}$. *Then any soft open cover* $\mathscr{L}=\{(V_i,P):i\in I\}\subset\tau_{\tilde{c}}$ *for* $(G,P)$ *has a Lebesgue soft element.*

**Proof** We assume that $\mathscr{L}$ is a soft open cover for $(G,P)$ such that it has no Lebesgue soft element. Then, for each $n\in\mathbb{N}$ and $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$, we can obtain $(H_n,P)\in S(G,P)$ with $\delta(H_n,P)\tilde{\prec}\frac{\tilde{c}}{n}$ such that $(H_n,P)\notin S\left(V_i,P\right)$, $\forall\left(V_i,P\right)\in\mathscr{L}$. Let us choose a soft element $\tilde{h}_n$ from each $(H_n,P)$. Since $(G,P)$ is sequentially compact, the sequence $\left\{\tilde{h}_n\right\}$ is in $(G,P)$ and has a convergent subsequence $\left\{\tilde{h}_{n_k}\right\}$ with $\tilde{h}_{n_k}\to\tilde{x}\tilde{\in}(G,P)$. But $(G,P)\tilde{\subset}\Cup_{i\in I}\left(V_i,P\right)$, so, there exists $i_o\in I$ such that $\tilde{x}\tilde{\in}\left(V_{i_o},P\right)$. Since $\left(V_{i_o},P\right)$ is soft open, we obtain $\Theta\tilde{\prec}\tilde{c}_1\tilde{\in}\tilde{U}$ satisfying $\tilde{x}\in B\left(\tilde{x},\tilde{c}_1\right)\subset SE\left(V_{i_o},P\right)$ and hence $\tilde{x}\tilde{\in}[B_{\tilde{c}_1},P]\tilde{\subset}\left(V_{i_o},P\right)$. Since $\tilde{h}_{n_k}\to\tilde{x}$ and $\tilde{c},\tilde{c}_1\tilde{\in}(C,P)^o$, we can obtain $i_{n_o}\in\mathbb{N}$ such that $d\left(\tilde{x},\tilde{h}_{i_{n_o}}\right)\tilde{\prec}\frac{\tilde{c}_1}{2}$ and $\frac{2\tilde{c}}{i_{n_o}}\tilde{\prec}\tilde{c}_1$. Now, $\left(H_{i_{n_o}},P\right)\tilde{\subset}(B_{\tilde{c}_1},P)\tilde{\subset}\left(V_{i_o},P\right)$ and this is a contradiction. That is, if $\tilde{z}\tilde{\in}\left(H_{i_{n_o}},P\right)$, $d\left(\tilde{z},\tilde{x}\right)\tilde{\preceq}d\left(\tilde{z},\tilde{h}_{i_{n_o}}\right)+d\left(\tilde{h}_{i_{n_o}},\tilde{x}\right)\tilde{\prec}\frac{\tilde{c}}{i_{n_o}}+\frac{\tilde{c}_1}{2}\tilde{\prec}\frac{\tilde{c}_1}{2}+\frac{\tilde{c}_1}{2}=\tilde{c}_1$. □

**Theorem 5.20** *Suppose that* $(\tilde{X}, d, P)$ *is a soft cone metric space that meets the condition* $(D4)$ *and* $\tau_{\tilde{c}}$ *is the elementary soft metric topology on* $\tilde{X}$. *Then any sequentially compact soft set* $(G, P) \in S(\tilde{X})$ *is compact.*

**Proof** We assume that $\mathscr{L} = \{(V_i, P) : i \in I\} \subset \tau_{\tilde{c}}$ is soft open cover for $(G, P)$. Since $(G, P)$ is sequentially compact, for any soft set $(H, P) \in S(G, P)$ with $\delta(H, P) \tilde{\prec} \tilde{c}$, there is $\Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}$ and there is $i_o \in I$ such that $(H, P) \tilde{\subset} (V_{i_o}, P)$. Since $(G, P)$ is totally bounded, then $(G, P) \tilde{\subset} \Cup_{i=1}^{n} (U_i, P)$, where $\delta(U_i, P) \tilde{\prec} \tilde{c}$. Thus, we can find $(V_1, P), (V_2, P), ..., (V_n, P) \in \mathscr{L}$ such that $(U_i, P) \tilde{\subset} (V_{i_o}, P)$. Namely, $(G, P) \tilde{\subset} \Cup_{i=1}^{n} (U_i, P) \tilde{\subset} \Cup_{i=1}^{n} (V_i, P)$ and hence $(G, P)$ is compact. □

**Definition 5.21** *[42] Assume that* $\left(\tilde{X}, \tau, P\right)$ *is an elementary soft topological space. We call* $\mathfrak{B}_{\tilde{x}} \subset \tau$ *containing any soft element* $\tilde{x} \tilde{\in} \tilde{X}$ *a soft local base at* $\tilde{x}$ *if there is* $(B_{\tilde{x}}, P) \in \mathfrak{B}_{\tilde{x}}$ *satisfying* $\tilde{x} \tilde{\in} (B_{\tilde{x}}, P) \tilde{\subset} (U, P)$ *for every soft open set* $(U, P) \in \tau$ *containing* $\tilde{x}$.

**Example 5.22** *If* $(\tilde{X}, d, P)$ *is a soft cone metric space that meet the condition* $(D4)$,

$$\mathfrak{B}_{\tilde{x}} = \left\{(B_{\tilde{c}}, P) : \tilde{x} \tilde{\in} \tilde{X}, \Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}\right\}$$

*is a soft local base at* $\tilde{x} \tilde{\in} \tilde{X}$ *for the elementary soft topology* $\tau_{\tilde{c}}$ *on* $\tilde{X}$.

**Theorem 5.23** *Every soft cone metric space* $(\tilde{X}, d, P)$ *that meets the condition* $(D4)$ *is first countable.*

**Proof** It is enough to prove that $\mathfrak{B}_{\tilde{x}} = \left\{(B_{\frac{\tilde{c}}{n}}, P) : n \in \mathbb{N}, \Theta \tilde{\prec} \tilde{c} \tilde{\in} \tilde{U}\right\}$ is a soft local base at $\tilde{x} \tilde{\in} \tilde{X}$. Assume that $(U, P) \in \tau_{\tilde{c}}$ is a soft open such that $\tilde{x} \tilde{\in} (U, P)$. Then we can obtain $\Theta \tilde{\prec} \tilde{c}_1 \tilde{\in} \tilde{U}$ satisfying $\tilde{x} \in B(\tilde{x}, \tilde{c}_1) \subset SE(U, P)$. By Lemma 3.10, we have $n_o \in \mathbb{N}$ such that $\frac{\tilde{c}}{n_o} \tilde{\prec} \tilde{c}_1$. Hence, $\tilde{x} \in B\left(\tilde{x}, \frac{\tilde{c}}{n_o}\right) \subset B(\tilde{x}, \tilde{c}_1) \subset SE(U, P)$. It implies that $\tilde{x} \tilde{\in} (B_{\frac{\tilde{c}}{n_0}}, P) \tilde{\subset} (U, P)$. Thus $\mathfrak{B}_{\tilde{x}}$ is a soft local base at $\tilde{x}$. That is, $(\tilde{X}, d, P)$ is first countable. □

**Proposition 5.24** *Suppose that* $(G, P) \in S(\tilde{X})$ *is a bounded soft set in* $(\tilde{X}, d, P)$ *that meets the condition* $(D4)$ *and* $(C, P)$ *is a strongly minihedral soft cone. Then* $\delta(G, P) = \delta\overline{(G, P)}$.

**Proof** $\delta(G, P) \tilde{\preceq} \delta\overline{(G, P)}$ since $(C, P)$ is a strongly minihedral soft cone and $(G, P) \tilde{\subset} \overline{(G, P)}$. Conversely, give $\tilde{x}, \tilde{y} \tilde{\in} \overline{(G, P)}$. There exist $\tilde{x}_n, \tilde{y}_n \tilde{\in} (G, P)$ with $\tilde{x}_n \to \tilde{x}$, $\tilde{y}_n \to \tilde{y}$ since $(\tilde{X}, d, P)$ is first countable. From the soft closedness of $(C, P)$ and by Lemma 5.13,

$$\lim_{n \to \infty} d(\tilde{x}_n, \tilde{y}_n) = d(\tilde{x}, \tilde{y}) \tilde{\preceq} \delta(G, P).$$

Since $\tilde{x}$ and $\tilde{y}$ are any two soft elements and $(C, P)$ is strongly minihedral, we finalize that $\delta\overline{(G, P)} \tilde{\preceq} \delta(G, P)$. □

**Definition 5.25** *Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space. We call a map* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *sequentially continuous if each sequence* $\{\tilde{x}_n\}$ *of soft elements in* $\tilde{X}$ *when* $\tilde{x}_n \to \tilde{x}$ *implies* $L\tilde{x}_n \to L\tilde{x}$.

**Definition 5.26** *Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space that meets the condition* $(D4)$. *We call a map* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *continuous at* $\tilde{x}\tilde{\in}\tilde{X}$ *if there is* $(U,P) \in \tau_{\tilde{c}}$ *containing* $\tilde{x}$ *such that* $L(SE(U,P)) \subset SE(V,P)$ *for each* $(V,P) \in \tau_{\tilde{c}}$ *containing* $L\tilde{x}$. *We call* $L$ *continuous on* $(\tilde{X}, d, P)$ *if it is continuous at every soft element of* $\tilde{X}$.

**Definition 5.27** *Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space. We call a map* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *a soft function if it meets the following condition* $(L_*)$.

$(L_*)$ *For* $a \in X$ *and each* $\alpha \in P$, $\left\{\tilde{x}(\alpha) : \tilde{x}(\alpha) = a,\ \tilde{x}\tilde{\in}\tilde{X}\right\}$ *is a singleton set.*

**Proposition 5.28** *Suppose that* $(\tilde{X}, d, P)$ *is a soft cone metric space that meets the condition* $(D4)$. *Then,* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *is a continuous mapping if and only if* $L$ *is sequentially continuous.*

**Proof** Let us suppose that $\Theta\tilde{\prec}\tilde{c}\tilde{\in}\tilde{U}$ and $\{\tilde{x}_n\}$ is a sequence of soft elements of $\tilde{X}$ as $\tilde{x}_n \to \tilde{x}\tilde{\in}\tilde{X}$. We obtain $\Theta\tilde{\prec}\tilde{c}_1\tilde{\in}\tilde{U}$ satisfying $L(B(\tilde{x}, \tilde{c}_1)) \subset B(L\tilde{x}, \tilde{c})$ since $L$ is continuous at $\tilde{x}\tilde{\in}\tilde{X}$. Since $\{\tilde{x}_n\}$ is convergent, we obtain $N \in \mathbb{N}$ satisfying $d(\tilde{x}_n, \tilde{x})\tilde{\prec}\tilde{c}_1$ and hence $d(L\tilde{x}_n, \tilde{x})\tilde{\prec}\tilde{c}$ for each $n \geq N$. The converse holds since $(\tilde{X}, d, P)$ is first countable. □

**Corollary 5.29** *Suppose that* $(\tilde{X}, d, P)$ *is a soft cone metric space that meets the condition* $(D4)$. *If* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *is continuous (sequentially continuous), then for any* $(G,P) \in S(\tilde{X})$, $L\left(\overline{SE(G,P)}\right) \subset \overline{L(SE(G,P))}$.

## 6. Fixed point theory

**Definition 6.1** *Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space and* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *is a mapping.*

1. *We call a soft element* $\tilde{x}_o\tilde{\in}\tilde{X}$ *that satisfies the condition* $L\tilde{x}_o = \tilde{x}_o$ *a fixed soft element of* $L$.
2. *We create a sequence* $\{\tilde{x}_n\}$ *of soft elements satisfying the relation*

$$\tilde{x}_0 = L\tilde{x}_1, \tilde{x}_1 = L\tilde{x}_2, ..., \tilde{x}_{n-1} = L\tilde{x}_n$$

   *for each* $\tilde{x}_o\tilde{\in}\tilde{X}$. *Then, we call* $\{\tilde{x}_n\}$ *created by iteration.*

3. *We call* $L$ *a contractive mapping on* $\tilde{X}$ *if there is a soft real number* $\tilde{t}$ *satisfying* $d(L\tilde{x}, L\tilde{y})\tilde{\preceq}\tilde{t}d(\tilde{x}, \tilde{y})$ *for each* $\tilde{x}, \tilde{y}\tilde{\in}\tilde{X}$, *where* $\bar{0}\tilde{\leq}\tilde{t}\tilde{<}\bar{1}$.

**Theorem 6.2** *[41] Assume that* $(\tilde{X}, d, P)$ *is a soft cone metric space. Then, a contractive mapping* $L$ *on* $\tilde{X}$ *has a unique fixed soft element in* $\tilde{X}$ *and the sequence* $\{L\tilde{x}_n\}$ *converges to the fixed soft element.*

**Definition 6.3** *Assume that* $(\tilde{X}, d, P)$ *is a complete soft cone metric space and* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *is a mapping.*

1. *We call the mapping* $L$ *diametrically contractive if for all soft closed bounded set* $(G,P) \in S(\tilde{X})$, $\delta((G',P)) \tilde{\prec} \delta(G,P)$ *such that* $\Theta \tilde{\prec} \delta(G,P)$, *where* $(G',P) = SS(L(SE(G,P)))$. *Obviously, any diametrically contractive mapping is contractive.*

2. *We call a soft set* $(G,P) \in S(\tilde{X})$ $L-$*invariant if* $L(SE(G,P)) \subset SE(G,P)$.

**Theorem 6.4** *Suppose that* $(\tilde{X},d,P)$ *is a sequentially compact soft cone metric space that meets the condition* $(D4)$ *and* $(C,P)$ *is a strongly minihedral soft cone. A mapping* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *has a fixed soft element if it is a diametrically contractive mapping.*

**Proof** Let us take the class

$$\mathscr{F} = \left\{ \ (G,P) \in S(\tilde{X}) : (G,P) \text{ is nonempty, sequentially compact and } L-\text{invariant} \ \right\}.$$

$\tilde{X} \in \mathscr{F}$ implies $\mathscr{F} \neq \varnothing$ since $\tilde{X}$ is sequentially compact and $L\left(SE(\tilde{X})\right) \subset SE(\tilde{X})$. $\mathscr{F}$ is partially ordered by $(G_1,P),(G_2,P) \in \mathscr{F}$, $(G_1,P) \tilde{\preceq} (G_2,P) \Leftrightarrow (G_1,P) \tilde{\subset} (G_2,P)$. Thus, each chain $\varsigma$ in $\mathscr{F}$ has the finite intersection property. The sequential compactness implies compactness since $(\tilde{X},d,P)$ that meets the condition $(D4)$. Then $(H,P) = \Cap\{(G,P) : (G,P) \in \varsigma\} \neq \Phi$ and $(H,P)$ is sequentially compact. Also

$$\begin{aligned} L(SE(H,P)) &= L((\cap\{SE(G,P) : (G,P) \in \varsigma\})) \\ &\subset \cap\{L(SE(G,P)) : (G,P) \in \varsigma\} \\ &\subset \cap\{SE(G,P) : (G,P) \in \varsigma\} \\ &= SE(H,P). \end{aligned}$$

Clearly, $(H,P)$ is a lower bound of $\mathscr{F}$. $\mathscr{F}$ has a minimal set element $(G,P)$ from Zorn's lemma. Let $\Upsilon_o = L(SE(G,P))$. Since the continuous image of a compact set is compact and $(G,P)$ is $L-$invariant, it is seen that $L(\Upsilon_o) \subset L(SE(G,P)) = \Upsilon_o$ and so $(G',P) = (G_o,P) = SS(\Upsilon_o)$ is compact, i.e. $(G_o,P) \in \mathscr{F}$. From minimality of $(G,P)$, we get $(G,P) = (G_o,P)$ and $(G,P) = (G',P)$. Therefore $\delta(G,P) = \delta((G',P))$. Since the mapping $L$ is diametrical contractive, we obtain $\delta(G,P) = \Theta$. Namely $SE(G,P)$ consists of exactly one soft element $\tilde{z}$. Since $(G,P)$ is also $L-$invariant, $L\tilde{z} = \tilde{z}$. □

**Definition 6.5** *Assume that* $(\tilde{X},d,P)$ *is a complete soft cone metric space. We call a map* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *asymptotically diametrically contractive if for all nonincreasing sequence* $\{(G_n,P)\}$ *of soft closed bounded sets of* $S(\tilde{X})$ *with* $\Theta \tilde{\prec} \delta_x(\{(G_n,P)\})$, $\delta_x(\{(G'_n,P)\}) \tilde{\prec} \delta_x(\{(G_n,P)\})$, *where*

$$\delta_x(\{(G_n,P)\}) = \lim_{n\to\infty} \delta(\{(G_n,P)\})$$

*is called the asymptotic diameter for* $\{(G_n,P)\}$.

Clearly, any asymptotically diametrically contractive mapping is diametrically contractive. Nevertheless, in sequentially compact soft cone metric space that meets the condition $(D4)$, the converse holds.

**Theorem 6.6** *Suppose that* $(\tilde{X}, d, P)$ *is a sequentially compact soft cone metric space that meets the condition* $(D4)$*. A map* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *is asymptotically diametrically contractive if it is a diametrically contractive mapping.*

**Proof** We assume that $\{(G_n, P)\}$ is a nonincreasing sequence of soft closed bounded set of $S(\tilde{X})$ satisfying

$$\Theta \tilde{\precsim} \delta_x(\{(G_n, P)\}) = \lim_{n\to\infty} \delta(\{(G_n, P)\}).$$

It follows that each $(G_n, P)$ is compact from the compactness of $(\tilde{X}, d, P)$. So, $(G'_n, P) = SS(L(\Upsilon_n))$ is also compact since $L$ is continuous. From compactness of $(G'_n, P)$ we can obtain $\tilde{x}_n, \tilde{y}_n \tilde{\in} (G_n, P)$ satisfying $d(L\tilde{x}_n, L\tilde{y}_n) = \delta((G'_n, P))$ from Proposition 5.14. Since $\{(G_n, P)\}$ is nonincreasing and $(G_1, P)$ is also compact, we can suppose that $\tilde{x}_n \to \tilde{x}$, $\tilde{y}_n \to \tilde{y}$, without losing generality. Thus, from Proposition 5.14 and from $L$ is continuous, we obtain

$$\delta_x(\{(G'_n, P)\}) = \lim_{n\to\infty} \delta(\{(G'_n, P)\}) = \lim_{n\to\infty} d(L\tilde{x}_n, L\tilde{y}_n) = d(L\tilde{x}, L\tilde{y}).$$

If $\tilde{x} = \tilde{y}$ then

$$\delta_x(\{(G'_n, P)\}) = d(L\tilde{x}, L\tilde{y}) = \Theta \tilde{\precsim} \delta_x(\{(G_n, P)\}).$$

If $\tilde{x} \neq \tilde{y}$ then

$$\begin{aligned} \delta_x(\{(G'_n, P)\}) &= d(L\tilde{x}, L\tilde{y}) \tilde{\prec} d(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(\tilde{x}_n, \tilde{y}_n) \\ &\tilde{\precsim} \lim_{n\to\infty} \sup\{d(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \tilde{\in} (G_n, P)\} \\ &= \lim_{n\to\infty} \delta(\{(G_n, P)\}) = \delta_x(\{(G_n, P)\}). \end{aligned}$$

□

**Theorem 6.7** *Suppose that* $(\tilde{X}, d, P)$ *is a complete soft cone metric space that meets the condition* $(D4)$ *and* $(C, P)$ *is strongly minihedral. A map* $L : SE(\tilde{X}) \to SE(\tilde{X})$ *has a unique fixed soft element* $\tilde{z} \tilde{\in} \tilde{X}$ *if it is an asymptotically diametrically contractive mapping. Also, If* $L$ *has a bounded orbit* $\{L^n \tilde{x}_o\}_{n=0}^{\infty}$ *for some* $\tilde{x}_o \tilde{\in} \tilde{X}$*, then* $\{L^n \tilde{x}\}_{n=0}^{\infty}$ *converges* $\tilde{z}$ *for each* $\tilde{x} \tilde{\in} \tilde{X}$*.*

**Proof** Since $L$ contractive, boundedness for one orbit $\{L^n \tilde{x}_o\}$ of $L$ gives boundedness of each orbit $\{L^n \tilde{x}\}$. Take $\Upsilon_n = \{L^m \tilde{x} : m \geq n\} = \{L^n \tilde{x}, L^{n+1} \tilde{x}, ...\}$. Note that $\Upsilon_n = L(\Upsilon_{n-1}), \forall n \geq 1$. Put $(G_n, P) = SS(\Upsilon_n)$, then $(G_n, P) = SS(L(\Upsilon_{n-1})) = (G'_{n-1}, P)$. From Theorem 3.5 and Proposition 5.14 and continuity of $L$, we have

$$\begin{aligned} \delta_x\left(\left\{\overline{(G_n, P)}\right\}\right) &= \delta_x(\{(G_n, P)\}) = \delta_x\left(\{(G'_{n-1}, P)\}\right) \\ &= \delta_x(\{(G'_n, P)\}) = \delta_x\left(\left\{\overline{(G'_n, P)}\right\}\right). \end{aligned}$$

We calculate that $\delta_x(\{(G_n, P)\}) = \Theta$ by asymptotically diametrical contractivity of $L$. Hence, $\{L^n \tilde{x}\}$ is Cauchy and thus convergent. Take $\lim_{n\to\infty} \{L^n \tilde{x}\} = \tilde{z}$. Since $L(L^n \tilde{x}) \to L\tilde{z}$, $L^{n+1}\tilde{x} \to \tilde{z}$ and $L\tilde{z} = \tilde{z}$. We obtain that the fixed soft element is unique from the contractivity of $L$. □

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