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GENERALIZATION OF STATISTICALLY CONVERGENT

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Abstract. In the late 1950's and early 1960's Kurzweil and Henstock presented the concept of Gauge integral. Following their results, Savas and Patterson extended this concept to summability theory by considering $f(\psi)$ real valued function which is integrable in the Gauge sense on $(1, \infty)$. The goal of this paper includes the extension of these notion to statistical convergence. This will be accomplished by presenting the definition of statistically convergent to L via cardinality in Lebesgue sense. Natural implications and variations are also presented.

В кінці 1950-х та на початку 1960-х років Курцвайль і Хенсток сформулювали концепцію калібрувального інтеграла. Савас і Паттерсон поширили це на теорію підсумовування, розглянувши дійсні функції $f(\psi)$, інтегровні в калібрувальному сенсі на $(1, \infty)$. Метою цієї роботи є поширення цього поняття на випадок статистичної збіжності. Для цього дається визначення статистичної збіжності за мірою Лебега. Обговорюються наслідки та можливі варіанти цього підходу.

1. Introduction, Preliminaries and Definitions

In 1957 Kurzweil [5] presented a new concept of integral which is called Gauge Integral. This notion allows us to extend the class of integrable functions beyond those of Lebesgue integrable. In [4] Henstock refined and placed this notion on a more solid foundation. Let us now present the definition of Gauge integral that was defined in [11].

Definition 1.1. [11] A tagged partition of an interval $I = [a, b]$ is a finite set or ordered pairs

$$D = \{(t_i, I_i) : 1 \leq i \leq m\}$$

where $\{I_i : 1 \leq i \leq m\}$ is a partition of I consisting of closed non overlapping subintervals and t_i is a point belonging to I_i; t_i is called the tag associated with I_i. If $f : I \to \mathbb{R}$, the *Riemann sum* of f with respect to D is defined to be

$$S(f, D) = \sum_{i=1}^{m} f(t_i)\ell(I_i),$$

where $\ell(I_i)$ is the length of the subinterval I_i. If $\delta : I \to (0, \infty)$ is a positive function, we define an open interval valued function on I by setting $\gamma(t) = (t - \delta(t), t + \delta(t))$. If $I_i = [x_i, x_{i+1}]$, we can write $t_i \in I_i \subset \gamma(t_i)$ instead of $t_i - \delta < x_i \leq t_i \leq x_{i+1} < t_i + \delta$. Any interval γ defined on I such that $\gamma(t)$ is an open interval containing t for each $t \in I$ is called a *Gauge* on I. Let us denote the set of all such interval by Δ_G. If $D = \{(t_i, I_i) : 1 \leq i \leq m\}$ is a tagged partition of I and γ is a Gauge on I, we say that D is $\gamma - fine$ if $t_i \in I_i \subset \gamma(t_i)$ is satisfied. Let $f : [a, b] \to \mathbb{R}$. If $f : [a, b] \to \mathbb{R}$. f is said to be Gauge integrable over $[a, b]$ if there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a Gauge γ on $[a, b]$ such that $|S(f, D) - A| < \varepsilon$ whenever D is a $\gamma - fine$ tagged partition of $[a, b]$. The number A is called the Gauge integral of f over $I = [a, b]$ and is

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denoted by $\int_a^b f$ or $\int_I f$; when we encounter integrals depending upon parameters, it is also convenient to write $\int_a^b f(t)$ or $\int_I f(t)$.

Throughout this paper we shall use the notion of bounded variation which is as follows: Let f be a function on $[a,b]$. Given a partition $P = \{[x_{k-1}, x_k]\}$ of $[a,b]$, the variation of f with respect to P is

$$V(f,P) = \sum_k |f(x_k) - f(x_{k-1})|,$$

and the variation of f over $[a,b]$ is

$$V_a^b f = \sup_P V(f,P),$$

where the supremum is taken over all partitions P of $[a,b]$. If $V_a^b f$ is finite, then f is said to be of bounded variation on $[a,b]$. The set of all such functions is denoted by $BV([a,b])$.
On the other hand, in 1951 Fast [2] introduced an extension the concept of sequential limit to statistically convergence which as follows:

Definition 1.2. If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$, then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \text{ and } \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly, $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$. A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to L if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero. In this case, we will denote statistically convergence as $st-\lim x_k$.

Following Fast's definition Schoenberg in [10] presented a bridge of this concept to summability theory. Recently, statistical convergence has been one of the most active areas in summability theory thanks to Fridy's presentation in [3] and many other papers were studied in this area (see [7], [8]). Afterward, strongly summable single valued functions were studied by Borwein in [1]. Following Borwein's work Nuray [6] extended his notion via λ-strongly summability and λ-statistically convergent functions by taking nonnegative real-valued Lebesgue measurable function on $(1,\infty)$. Prior to present Nuray's notions, let us note that the following definition.

Definition 1.3. [6] Let $\lambda = (\lambda_n)$ be non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. Δ denote the set of all such sequences. For a sequence $x = (x_n)$ the generalized de la Vallée Poussin mean is defined by

$$t_n(x) = \frac{1}{n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

Definition 1.4. [6] Let $\lambda \in \Delta$ and $f(\psi)$ be a real valued function which is measurable in the Lebesgue sense in the interval $(1,\infty)$, if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \int_{n-\lambda_n+1}^{n} |f(\psi) - L|\, d\psi = 0,$$

then we say that the function $f(\psi)$ is $\lambda-$strongly summable to L. In this case we write $[W,\lambda]-\lim f(\psi)=L$ or $f(\psi)\rightarrow L\,[W,\lambda]$. If we take $\lambda_n=n$, then $[W,\lambda]$ reduced to $[W]$, the space of all all strongly double summable functions.

Definition 1.5. [6] Let $\lambda\in\Delta$ and $f(\psi)$ be a real-valued function which is measurable on $(1,\infty)$, if for every $\varepsilon>0$,

$$\lim_{n\rightarrow\infty}\frac{1}{\lambda_n}\left|\left\{\psi\in I_n:\left|f(\psi)-L\right|\geq\varepsilon\right\}\right|=0,$$

then we say that the function $f(\psi)$ is $\lambda-$statistically convergent to L, where the vertical bars indicate the Lebesgue measurable of the enclosed set. The space of all statistical convergence functions will be denoted by (S_f,λ). In this case, we write $[S_f,\lambda]-\lim f(\psi)=L$ or $f(\psi)\rightarrow L\,[S_f,\lambda]$.

The following is an example of such convergence.

Example 1.6. Let us consider a function $f(\psi)$ which is defined by

$$f(\psi)=\begin{cases}\psi, n-\frac{1}{\log\lambda_n}+\frac{1}{\lambda_n}+1\leq\psi\leq n,\\ 0,\text{otherwise,}\end{cases}$$

for every $\varepsilon>0$,

$$\lim_{n\rightarrow\infty}\frac{1}{\lambda_n}\left|\left\{\psi\in I_n:\left|f(\psi)-0\right|\geq\varepsilon\right\}\right|=\lim_{n\rightarrow\infty}\frac{\frac{1}{\log\lambda_n}+\frac{1}{\lambda_n}}{\lambda_n}=0,$$

i.e., $[S_f,\lambda]-\lim f(\psi)=0$.

In addition to these definitions, please note the following theorem in [6].

Theorem 1.7. [6] *Let $\lambda\in\Delta$ and $f(\psi)$ be a real valued function which is measurable in the Lebesgue sense in the interval $(1,\infty)$, then $[W,\lambda]\subset[S_f,\lambda]$ and the inclusion is proper.*

In 2019, Savas and Patterson in [9] introduced the new concept of strongly Cesáro type summability theory by considering Gauge integral and the following definition:

Definition 1.8. [9] Let us consider $\delta: I_i=(t_i-\delta(t_i),t_i+\delta(t_i)]\rightarrow(0,\infty)$ is a positive function, and $[a,b]=\cup I_i$ with $-\infty<a<b<\infty$. We define an open interval valued function on I by setting $\overline{\gamma}=\overline{\gamma}(t_i)=(t_i-\delta(t_i),t_i+\delta(t_i))$. If $J_i=[i-\lambda_i+1,i]$, we can write $t_i\in J_i\subset\overline{\gamma}(t_i)$ instad of $t_i-\delta(t_i)<i-\lambda_i+1\leq t_i\leq i<t_i+\delta(t_i)$. Let $\overline{\gamma}=\overline{\gamma}(t_i)\in\Delta_G$, and let $f(\psi)$ be a real valued function which is measurable Gauge sense in the interval $(1,\infty)$. Provided that $\int f(\psi)$ and $\int|f(\psi)|$ exist in the gauge sense and

$$\lim_{t_i\rightarrow\infty}\frac{1}{\xi(t_i)}\int_{t_i-\delta(t_i)}^{t_i+\delta(t_i)}\left|f(\psi)-L\right|d\psi=0,$$

where $\xi(t_i)=(t_i+\delta(t_i))-(t_i-\delta(t_i))=2\delta(t_i)$, then we say that the function $f(\psi)$ is $\overline{\gamma}-$strongly summable to L with respect Gauge. In this case, we write $[G,\overline{\gamma}]-\lim f(\psi)=L$ or $f(\psi)\rightarrow L\,[G,\overline{\gamma}]$.

Using the definitions above, Savas and Patterson also established the following theorem which grants us a connection between strongly summability in the Lebesgue sense and in the Gauge sense.

Theorem 1.9. [9] *Let $\lambda=(\lambda_n)\in\Delta$, $\overline{\gamma}=\overline{\gamma}(t_i)\in\Delta_G$, $I_i=[t_i-\delta(t_i),t_i+\delta(t_i)]$ and $[a,b]=\cup I_i$ with $-\infty<a<b<\infty$, and $f(\psi)$ be a real valued function in the Gauge sense in the interval $(1,\infty)$, then*

(1) $[W,\lambda]\subset[G,\overline{\gamma}]$

(2) If $f(\psi)$ is bounded variation and f is $\overline{\gamma}-$strongly summable to L with respect to Gauge sense over every measurable subset of $[t_i - \delta(t_i), t_i + \delta(t_i)]$ (*i.e.*, if $C_E f$ is Gauge integrable over $[t_i - \delta(t_i), t_i + \delta(t_i)]$) for every measurable $E \subset t_i - \delta(t_i), t_i + \delta(t_i))$, then f is $[W] - \lim f(\psi) = L$.

2. Main Results

We begin this section with the following new definition.

Definition 2.1. Let $\lambda \in \Delta$ and $f(\psi)$ be a real-valued function in the interval $(1, \infty)$, for every $\varepsilon > 0$, let $A = \{\psi \in I_n : |f(\psi) - L| \geq \varepsilon\}$, $\{A_i : i \in \mathbb{N}\}$ be a countable partition of A, and $\alpha_i = \sup\{\psi \in A_i\}$. Provided that

$$\lim_{n\to\infty} \frac{1}{\alpha_n} |\{\psi \leq \alpha_i : |f(\psi) - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the Lebesgue measure of the enclosed set, then we say $f(\psi)$ is statistically convergent to L via cardinality. In this case, we write $S_f^* - \lim f(\psi) = L$ or $f(\psi) \to L \left[S_f^*\right]$. The class of the $\lambda-$statistically convergent to L via cardinality is denoted by $\left[S_f^*\right]$.

This following are examples of a measurable and non-measurable functions, respectively that satisfy Definition 2.1.

Example 2.2. $f(\psi)$ be a real-valued function which is measurable on $(1, \infty)$. Define by

$$f(\psi) = \begin{cases} 1 & \text{if } \psi \text{ is a square } / \{1\}, \\ 0 & \text{if } \psi \in (1, \infty) / \psi \text{ is not a square.} \end{cases}$$

Example 2.3. Let S a non-measurable subset of $(1, \infty)$. Define a function $f(\psi)$ by

$$f(\psi) = \begin{cases} 1 & \text{if } \psi \in S \cup (\psi \text{ is an even square}), \\ 0 & \text{if } \psi \in S \cup (\psi \text{ is an odd square}), \\ 0 & \text{if otherwise.} \end{cases}$$

Let us consider the following inclusion theorems.

Theorem 2.4. *If* $\lim_{n\to\infty} \inf \frac{\lambda_n}{\alpha_n} > 0$ *and* $\frac{\lambda_n}{\alpha_n} = O(1)$, *then* $[S_f, \lambda] \subseteq \left[S_f^*\right]$.

Proof. Let $\varepsilon > 0$ and $[S_f, \lambda] - \lim f(\psi) = L$. We write

$$\{\psi \leq \alpha_i : |f(\psi) - L| \geq \varepsilon\} \supset \{\psi \in I_n : |f(\psi) - L| \geq \varepsilon\}.$$

Therefore,

$$\begin{aligned} \frac{1}{\alpha_n} |\{\psi \leq \alpha_i : |f(\psi) - L| \geq \varepsilon\}| &\geq \frac{1}{\alpha_n} |\{\psi \in I_n : |f(\psi) - L| \geq \varepsilon\}| \\ &\geq \frac{\lambda_n}{\alpha_n} . \frac{1}{\lambda_n} |\{\psi \in I_n : |f(\psi) - L| \geq \varepsilon\}|. \end{aligned}$$

Hence by using $\lim_{n\to\infty} \inf \frac{\lambda_n}{\alpha_n} > 0$ and taking the limit $n \to \infty$ we get $f(\psi) \to L \, [S_f, \lambda]$ implies $f(\psi) \to L \left[S_f^*\right]$. ☐

Theorem 2.5. $[W, \lambda] \subset \left[S_f^*\right]$ *and for the condition* $\lim_{n\to\infty} \inf \frac{\alpha_n}{\lambda_n} > 1$, *the inclusion is proper.*

Proof. Let $\varepsilon > 0$ and $[W,\lambda] - \lim f(\psi) = L$. We write

$$\int\limits_{\psi \in I_n} |f(\psi) - L|\, d\psi = \int\limits_{\{\psi :\, \psi \le \alpha_i\}} |f(\psi) - L|\, d\psi \ge \varepsilon \left\{ \psi \le \alpha_i : |f(\psi) - L| \ge \varepsilon \right\}.$$

Therefore, $[W,\lambda] - \lim f(\psi) = L$ implies $S_f^* - \lim f(\psi) = L$. Let us consider the following function

$$f(\psi) = \begin{cases} \psi, & n - \ln(\lambda_n) + 1 \le \psi \le n, \\ 0 & \text{otherwise} \end{cases}$$

$f(\psi)$ is not bounded function, for every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{\alpha_n} \left| \{ \psi \le \alpha_i : |f(\psi) - L| \ge \varepsilon \} \right| \\ \ge \; & \lim_{n\to\infty} \frac{1}{\alpha_n} \frac{\alpha_n}{\lambda_n} \left| \{ \psi \le \alpha_i : |f(\psi) - L| \ge \varepsilon \} \right| \\ = \; & \lim_{n\to\infty} \frac{\ln(\lambda_n)}{\lambda_n} = 0, \end{aligned}$$

i.e., $S_f^* - \lim f(\psi) = 0$. However,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \int\limits_{n-\lambda_n+1}^{n} |f(\psi) - 0|\, d\psi = \infty,$$

i.e., $f(\psi) \nrightarrow L\, [W,\lambda]$. Hence, the inclusion is proper. □

Theorem 2.6. $[G, \overline{\gamma}] \subsetneq \left[S_f^* \right]$.

Proof. Suppose that $\varepsilon > 0$, $[G, \overline{\gamma}] - \lim f(\psi) = L$. Therefore, we can obtain the following

$$\begin{aligned} \int\limits_{\psi \in \overline{\gamma}(t_i)} |f(\psi) - L|\, d\psi & \ge \int\limits_{\{\psi \in \overline{\gamma}(t_i)\, :\, |f(\psi) - L| \ge \varepsilon\}} |f(\psi) - L|\, d\psi \\ & \ge \varepsilon \left| \{ \psi \in \overline{\gamma}(t_i) : |f(\psi) - L| \ge \varepsilon \} \right| \\ & \gneqq \varepsilon \left| \{ \psi \le \alpha_i : |f(\psi) - L| \ge \varepsilon \} \right| \end{aligned}$$

which implies that $f(\psi) \nrightarrow L\, [S_f, \lambda]$. □

Theorem 2.7. *If* $\lim_{n\to\infty} \inf \frac{\overline{\gamma}(t_i)}{\alpha_n} > 0$ *and* $f(\psi)$ *is a bounded variation, then* $\left[S_f^* \right] \subseteq [G, \overline{\gamma}]$.

Proof. Suppose that $\left[S_f^* \right] - \lim f(\psi) = L$ and since $f(\psi)$ be a bounded variation, $f(\psi)$ will be a bounded function, and we say that $|f(\psi) - L| \le M$ for all ψ. Given $\varepsilon > 0$, we have that

$$\begin{aligned} \frac{1}{\overline{\gamma}(t_i)} \int\limits_{\psi \in \overline{\gamma}(t_i)} |f(\psi) - L|\, d\psi & = \frac{1}{\overline{\gamma}(t_i)} \int\limits_{\{\psi \in \overline{\gamma}(t_i)\, :\, |f(\psi) - L| \ge \varepsilon\}} |f(\psi) - L|\, d\psi \\ & \quad + \frac{1}{\overline{\gamma}(t_i)} \int\limits_{\{\psi \in \overline{\gamma}(t_i)\, :\, |f(\psi) - L| < \varepsilon\}} |f(\psi) - L|\, dx \\ & \le \frac{M}{\overline{\gamma}(t_i)} \left| \{ \psi \in \overline{\gamma}(t_i) : |f(\psi) - 0| \ge \varepsilon \} \right| + \varepsilon \\ & \le \frac{M}{\overline{\gamma}(t_i)} \left| \{ \psi \le \alpha_i : |f(\psi) - L| \ge \varepsilon \} \right| + \varepsilon. \\ & \le \frac{\overline{\gamma}(t_i)}{\alpha_n} \frac{M}{\overline{\gamma}(t_i)} \left| \{ \psi \le \alpha_i : |f(\psi) - L| \ge \varepsilon \} \right| + \varepsilon. \end{aligned}$$

Hence, $[G,\overline{\gamma}] - \lim f(\psi) = L$. □

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References

[1] D. Borwein, Linear Functionals with Strongly Cesáro Summability, Journal of London Mat. Soc., 40 (1965), 628–634.
[2] H. Fast, Sur la Convergence Statistique, Colloquium Mathematicum, 2 (1951), 241–244.
[3] J. A. Fridy, On Statistically Convergence, Analysis, 5 (1985), 301–313.
[4] R. Henstock, Definitions of Riemann Type of Variational Integral, Proc. London Math. Soc., 11 (1961), 402–418.
[5] J. Kurzweil, Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter, Czech. Math. J., 82 (1957), 418–449.
[6] F. Nuray, $\lambda-$Strongly Summable and $\lambda-$Statistically Convergent Functions, Iranian Journal of Sci. and Tech., 34 (4) (2010), 335–338.
[7] K. Raj, A. Choudhary and C. Sharma, Almost strongly Orlicz double sequence spaces of regular matrices and their applications to statistical convergence, Asian-Eur. J. Math., Vol.11 No.5 (2018), 1850073, (14pages).doi.org/10.1142/S1793557118500730.
[8] K. Raj and S. Jamwal, On some generalized statistical convergent sequence spaces, Kuwait J. Sci., 42 (3) (2015), 86–104.
[9] R. Savas and R. F. Patterson, Gauge Strongly Summability for Measurable Functions, Carpathian Journal of Mathematics, (accepted-preprint)
[10] I. J. Schoenberg, The Integrability of Certain Functions and Related Summability Theory, Am. Math. Month., 66 (1959), 361–375.
[11] C. Swartz, Introduction to Gauge Integrals, World Scientific Publishing Co., 1938.

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