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A generalization of the Alexander polynomial as an application of the delta derivative

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Abstract: In this paper, we define the delta derivative in the integer group ring and we show that the delta derivative is well defined on the free groups. We also define a polynomial invariant of oriented knot and link by carrying the delta derivative to the link group. Since the delta derivative is a generalization of the free derivative, this polynomial invariant called the delta polynomial is a generalization of the Alexander polynomial. In addition, we present a new polynomial called the difference polynomial of oriented knot and link, which is similar to the Alexander polynomial and is a special case of the delta polynomial.

Key words: Time scales, delta derivative, derivative in group rings, free derivative, Alexander polynomial

1. Introduction

We define here the delta (or Hilger) derivative in the integer group ring of an arbitrary group and we present a generalization of the Alexander polynomial of knots and links in S^3. The Alexander polynomial of the oriented link is a Laurent polynomial associated with the link in an invariant way. This polynomial was first defined by Alexander [3]. There are several ways to calculate the Alexander polynomial. One of them is the free derivative defined by Fox [7, 8]. The delta derivative is defined as a differential calculus on time scales (or measure chains) by Aulbach and Hilger [4, 9, 10].

The plan of this paper is as follows: Section 2 gives summary information about the free derivative, the Jacobian matrix, and knot group, respectively. In this section we also summarize how to calculate the Alexander polynomial from the Jacobian matrix for a knot group and we give some of its results. In Section 3, we describe the delta derivative and we give some of its results. We briefly explain the relation between the free derivative and the delta derivative in that the free derivative is a special case of the delta derivative in mathematical analysis. In Section 4, we define the delta derivative on an integer group ring and we show that the delta derivative is well defined on the free group. In Section 5, we define the delta polynomial by using information in Sections 3 and 4 similarly to Section 2 and we present the delta polynomial as a general case of the Alexander polynomial and a new polynomial called the difference polynomial. In the last part of this section, we prove that the delta polynomial is a knot invariant.

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2. Free differential calculus and the Alexander polynomial

Let G be an arbitrary group and $\mathbb{Z}G$ the integer group ring of G. A derivative in $\mathbb{Z}G$ is additive homomorphism $\nabla : \mathbb{Z}G \to \mathbb{Z}G$ such that

$$\nabla(xy) = \nabla(x) + x\nabla(y) \tag{1}$$

for any $x, y \in G$. The mapping ∇ is called a derivative of $\mathbb{Z}G$. The set of all derivatives in $\mathbb{Z}G$ can be thought of as a (left) $\mathbb{Z}G$-module in a natural manner. The following lemma contains some results of this derivative.

Lemma 2.1 *Let ∇ be a derivative.*

1. $\nabla(m) = 0$ *for* $m \in \mathbb{Z}$.
2. $\nabla(x^{-1}) = -x^{-1}\nabla(x)$.
3. $\nabla(x^n) = (1 + x + \cdots + x^{n-1})\nabla(x)$, *for* $n \geq 1$.
4. $\nabla(x^{-n}) = (x^{-1} + x^{-2} + \cdots + x^{-n})\nabla(x)$, *for* $n \geq 1$.

As for free groups, the structure of this module is quite clear [8].

Lemma 2.2 *If F_n is a free group generated by $x_1, x_2, \ldots, x_n$ and w is a word in F_n, then there are the following properties satisfied by the partial derivative of the free derivative $\frac{\partial}{\partial x_i} : F_n \to F_n$, see [8].*

1. $\dfrac{\partial(w_1 w_2)}{\partial x_i} = \dfrac{\partial w_1}{\partial x_i} + w_1 \dfrac{\partial w_2}{\partial x_i}$.
2. $\dfrac{\partial x_j}{\partial x_i} = \delta_{ij}$, *where* δ_{ij} *the Kronocker symbol* .
3. $\dfrac{\partial w^{-1}}{\partial x_i} = -w^{-1}\dfrac{\partial w}{\partial x_i}$.
4. $\dfrac{\partial x^n}{\partial x} = 1 + x + \cdots + x^{n-1}$, *for* $n \geq 1$.
5. $\dfrac{\partial x^{-n}}{\partial x} = -(x^{-1} + x^{-2} + \cdots + x^{-n})$, *for* $n \geq 1$.

Let $G = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n \rangle$ be a finitely presented group. Regarding the relations $r_1, r_2, \ldots, r_n$ as words in the x_j's, we form the Jacobian matrix $J = (\frac{\partial r_i}{\partial x_j})$, where these derivatives can be simplified by using relations in G. Let $J^\phi = (\frac{\partial r_i}{\partial x_j})^\phi$ denote the image of the Jacobian under the abelianization map $\phi : \mathbb{Z}G \to \mathbb{Z}[t, t^{-1}]$, sending each x_i to t. The matrix J^ϕ is the ϕ-Jacobian matrix or Alexander matrix of G. For details, see [7].

A link K with k components is a subset of $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$, consisting of k disjoint piecewise simple closed curves and a knot is a link with one component. In fact, two knots (or links) in $\mathbb{R}^3$ can be deformed continuously one into the other if and only if any diagram of one knot can be transformed into a diagram for the knot via a sequence of the Reidemeister moves formed in Figure 1. The equivalence relation on diagrams that is generated by all the Reidemeister moves is called ambient isotopy.

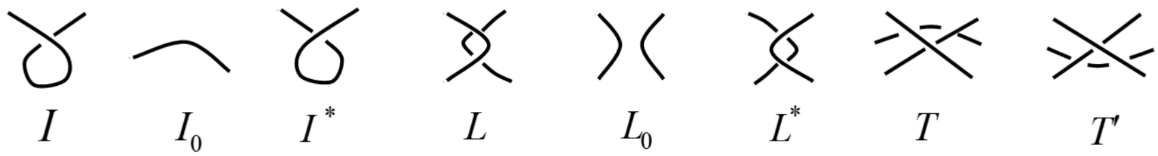

Figure 1. The Reidemeister moves. The first Reidemeister move: $I \leftrightarrow I_0$ or $I^* \leftrightarrow I_0$; The second Reidemeister move: $L \leftrightarrow L_0$ or $L^* \leftrightarrow L_0$; The third Reidemeister move: $T \leftrightarrow T'$.

Let K be a knot. The fundamental group $\pi_1(\mathbb{R}^3 \setminus K)$ of complement is called, simply, the knot group of K. We now assume that G presents the knot group $\pi_1(S^3 \setminus K)$. Then $G \setminus [G,G] \cong \langle t \rangle \cong \mathbb{Z}$. It is easy to see this if G is a Wirtinger presentation [15]. In this case, we can regard J^ϕ as having entries in the ring $\mathbb{Z}[t,t^{-1}]$ along with its subring $\mathbb{Z}[t]$ having the property that any finite set of elements has a greatest common divider (GCD). Any integer domain with this property is called a GCD domain. For more information, see [7].

We consider the ϕ-Jacobian matrix J^ϕ for a knot group $\pi_1(S^3 \setminus K)$ with respect to a presentation $G = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n \rangle$. Let E be the ideal generated by the $(n-1) \times (n-1)$ minors of J^ϕ. In [7], E is shown to be a nonzero principal ideal. The Alexander polynomial $\nabla_K(t)$ is, up to multiplying by any power $\pm t^k$, $k \in \mathbb{Z}$, a generator (i.e. a GCD) of E. If $\nabla_{K_1}(t)$ and $\nabla_{K_2}(t)$ are polynomials that are equal up to such a factor, we write $\nabla_{K_1}(t) \doteq \nabla_{K_2}(t)$. If $G = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n \rangle$ is a Wirtinger presentation of the knot group, then any one of the $(n-1) \times (n-1)$ minors of J^ϕ can be taken to be $\nabla_K(t)$; see [15].

The following lemma contains several important properties of the Alexander polynomial.

Lemma 2.3

1. *Let K be a knot; then $\nabla_K(t)$, is a symmetric Laurent polynomial, i.e.*

$$\nabla_K(t) = a_{-n}t^{-n} + a_{-(n-1)}t^{-(n-1)} + \ldots + a_{n-1}t^{n-1} + a_n t^n$$

   and

$$a_{-n} = a_n,\ a_{-(n-1)} = a_{n-1}, \ldots, a_{-1} = a_1.$$

2. *If K is a knot, then $\nabla_K(1) = 1$.*

3. *If K^* is the mirror image of K, then $\nabla_{K^*}(t) = \nabla_K(t)$.*

4. *If K is a trivial knot, then $\nabla_K(t) = 1$.*

5. *If K is a trivial μ-component ($\mu \geq 2$) link, then $\nabla_K(t) = 0$.*

For proof, see [15].

3. Delta (or Hilger) derivative

In recent years, a calculus on time scales (or measure chains) has been developed by several authors with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case) [1, 2, 4, 5, 9–12].

Nonempty closed subsets of the real numbers are considered to be time scales in for example [4, 5]. Moreover, regarding times scales, see [9, 10] for a discussion in the more general framework of measure chains. Let T be a time scale. We define the right jump function $\sigma : T \to T$ by $\sigma(t) = inf\{s \in T | s > t\}$ (supplemented by $inf\varnothing = supT$) and the left jump function $\rho : T \to T$ by $\rho(t) = sup\{s \in T | s < t\}$ (supplemented by $sup\varnothing = infT$). The graininess (or step-size) function $\mu : T \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for each $t \in T$. A point $t \in T$ is called right scattered if $\mu(t) > 0$ while the terminology right dense is used in the case of $\mu(t) = 0$.

The delta derivative defined by Aulbach and Hilger [4, 9, 10] is the usual derivative if $T = \mathbb{R}$ and the forward difference if $T = \mathbb{Z}$. In order to define the delta derivative of a function, we say that a subset U of T is open in T if it is open in the relative topology [13], i.e. if $U = V \cap T$ for some open set V in $\mathbb{R}$. A neighbourhood U of a point $t \in T$ is a subset of T that is open in T and contains t. A function f is said to be delta differentiable at a point $t \in T^i$ (where T^i denotes the set of points of T except for a left scattered maximal point) if f is defined at $\sigma(t)$, f is defined in a neighbourhood U of t, and there exists a quantity $f^{\triangle}(t)$, called the delta derivative of f at t, such that for each positive real number ε there exists a neighbourhood N of t such that $N \subseteq U$ and

$$|f(\sigma(t)) - f(s) - f^{\triangle}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for every $s \in N$. The following lemma contains results for this derivative.

Lemma 3.1 *Let $f, g : T \to \mathbb{R}$ and $t \in T^i$. Then the following hold [4, 14]:*

1. *If f is defined on $\mathbb{R}$ and differentiable at right dense point $t \in T^i$, then f is delta differentiable at t with*

$$f^{\triangle}(t) = f'(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s},$$

   where $s \in T \setminus \{\sigma(t)\}$.

2. *If $f^{\triangle}(t)$ exists, then f is continuous at t.*

3. *If $f^{\triangle}(t)$ exists, then $f(\sigma(t)) = f(t) + \mu(t) f^{\triangle}(t)$.*

4. *If t is right-scattered and f is continuous at t, then*

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

5. *If $f^{\triangle}(t)$, $g^{\triangle}(t)$ exists, and $(f + g)(t)$ is defined, then*

$$(f + g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t).$$

6. *If $f^{\triangle}(t)$ exists and λ is a constant, then*

$$(\lambda f)^{\triangle}(t) = \lambda f^{\triangle}(t).$$

7. *If* $f^{\triangle}(t)$, $g^{\triangle}(t)$ *exists, and* $(fg)(t)$ *is defined, then*

$$(fg)^{\triangle}(t) = f^{\triangle}(t)g(\sigma(t)) + f(t)g^{\triangle}(t).$$

8. *If* $f^{\triangle}(t)$ *exists on* T^i *and* f *is invertible on* T, *then*

$$(f^{-1})^{\triangle}(t) = -(f(\sigma(t)))^{-1} f^{\triangle}(t)f^{-1}(t)$$

on T^i.

9. *If* $f^{\triangle}(t) = 0$ *on* T^i, *then* f *is a constant on* T.

If the free derivative is defined on $\mathbb{Z}$ (or a subset of $\mathbb{R}$ that satisfies the properties of a ring), from equality (1) and property 1 of Lemma 3.1 we can write this derivative as

$$\frac{f(1) - f(t)}{1 - t}, \tag{2}$$

where the function f is free differentiable at a point $t \in \mathbb{Z} \setminus \{1\}$, f is defined at $\sigma(t)$, and f is defined in a neighborhood U of t. The derivative (2) is a special case of property 4 of Lemma 3.1. Hence, the delta derivative is a generalization of the free derivative.

4. Delta derivative in the group rings

We can now define the delta derivative on an integer group ring as follows. We shall write $\mathcal{D}$ for $f^{\triangle}$.

Definition 4.1 *Let* G *be an arbitrary group and* $\mathbb{Z}G$ *the integer group ring of* G. *The delta derivative* $\mathcal{D}$ *in* $\mathbb{Z}G$ *is additive homomorphism* $\mathcal{D} : \mathbb{Z}G \to \mathbb{Z}G$ *such that*

$$\mathcal{D}(xy) = \mathcal{D}(x)\sigma(y) + x\mathcal{D}(y) \tag{3}$$

for any $x, y \in \mathbb{Z}G$, *where* $\sigma : \mathbb{Z}G \to \mathbb{Z}$ *is the augmentation homomorphism. (Let* $\sigma_1 : \mathbb{Z}G \to \mathbb{Z}$ *be a augmentation map and* $\sigma_2 : \mathbb{Z} \to \mathbb{Z}$ *be a right jump map; then* $\sigma = \sigma_2 \circ \sigma_1 : \mathbb{Z}G \to \mathbb{Z}$ *is a augmentation map). The set of all delta derivatives in* $\mathbb{Z}G$ *can be thought of as a* $\mathbb{Z}G$*-module in a natural manner.*

Since $\mathcal{D}(x)$ in $\mathbb{Z}G$ is an additive homomorphism, it is a linear mapping. Linearity and the product rule (i.e. equality (3)) imply uniqueness. For example, since $(xy)z = x(yz)$ for $x, y, z \in \mathbb{Z}G$, $\mathcal{D}((xy)z) = \mathcal{D}(x(yz))$. In fact,

$$\begin{aligned}
\mathcal{D}((xy)z) &= \mathcal{D}(xy)\sigma(z) + xy\mathcal{D}(z) \\
&= (\mathcal{D}(x)\sigma(y) + x\mathcal{D}(y))\sigma(z) + xy\mathcal{D}(z) \\
&= \mathcal{D}(x)\sigma(y)\sigma(z) + x\mathcal{D}(y)\sigma(z) + xy\mathcal{D}(z)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}(x(yz)) &= \mathcal{D}(x)\sigma(yz) + x\mathcal{D}(yz) \\
&= \mathcal{D}(x)\sigma(yz) + x(\mathcal{D}(y)\sigma(z) + y\mathcal{D}(z)) \\
&= \mathcal{D}(x)\sigma(yz) + x\mathcal{D}(y)\sigma(z) + xy\mathcal{D}(z).
\end{aligned}$$

Since $\sigma : \mathbb{Z}G \to \mathbb{Z}$ is a homomorphism, $\sigma(yz) = \sigma(y)\sigma(z)$ and hence $\mathcal{D}((xy)z) = \mathcal{D}(x(yz))$. The following lemma contains some results of this derivative.

Lemma 4.2 *If $\mathcal{D}$ exists on $\mathbb{Z}G$, then*

1. $\mathcal{D}(m) = 0$ *for* $m \in \mathbb{Z}$.
2. $\mathcal{D}(x^{-1}) = -x^{-1}\sigma(x^{-1})\mathcal{D}(x)$.
3. $\mathcal{D}(x^n) = \left(\sigma(x^{n-1}) + \sigma(x^{n-2})x + \cdots + \sigma(x)x^{n-2} + x^{n-1}\right)\mathcal{D}(x)$.
4. $$\mathcal{D}(x^{-n}) = -\Big(\sigma(x^{-n})x^{-1} + \sigma(x^{-(n-1)})x^{-2} + \cdots + \sigma(x^{-2})x^{-(n-1)} \\ + \sigma(x^{-1})x^{-n}\Big)\mathcal{D}(x), \text{ for } n \geq 1.$$

Proof Proof follows from Lemma 3.1 and Definition 4.1 by simple calculations. □

Now we show that $\mathcal{D}$ is well defined on a free group.

Proposition 4.3 *Let F_n be a free group generated by $x_1, x_2, \ldots, x_n$ and w_i be arbitrary words in $\mathbb{Z}F_n$. There is a uniquely determined derivative $\mathcal{D} : \mathbb{Z}F_n \to \mathbb{Z}F_n$ with $\mathcal{D}(x_i) = w_i$.*

Proof $\mathcal{D}(x^{-1}) = -x^{-1}\sigma(x)^{-1}w_i$ follows from $\mathcal{D}(1) = 0$ and the product rule. Linearity and the product rule imply uniqueness. By defining $\mathcal{D}\left(x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\ldots x_{i_k}^{\varepsilon_k}\right)$ and using the product rule:

$$\mathcal{D}\left(x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\ldots x_{i_k}^{\varepsilon_k}\right) = \sigma\left(x_{i_2}^{\varepsilon_2}\ldots x_{i_k}^{\varepsilon_k}\right)\mathcal{D}\left(x_{i_1}^{\varepsilon_1}\right) + x_{i_1}^{\varepsilon_1}\sigma\left(x_{i_3}^{\varepsilon_3}\ldots x_{i_k}^{\varepsilon_k}\right)\mathcal{D}\left(x_{i_2}^{\varepsilon_2}\right) + \ldots \\ + x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\ldots x_{i_{k-1}}^{\varepsilon_{k-1}}\mathcal{D}\left(x_{i_k}^{\varepsilon_k}\right), \quad \varepsilon_i = \pm 1.$$

Then the product rule follows for combined words $w = uv$, $\mathcal{D}(w) = \mathcal{D}(u)\sigma(v) + u\mathcal{D}(v)$. The equation

$$\begin{aligned}\mathcal{D}\left(ux_i^{\varepsilon}x_i^{-\varepsilon}v\right) &= \mathcal{D}(u)\,\sigma\left(x_i^{\varepsilon}x_i^{-\varepsilon}v\right) + u\mathcal{D}\left(x_i^{\varepsilon}\right)\sigma\left(x_i^{-\varepsilon}v\right) + ux_i^{\varepsilon}\mathcal{D}\left(x_i^{-\varepsilon}\right)\sigma(v) \\ &\quad + ux_i^{\varepsilon}x_i^{-\varepsilon}\mathcal{D}(v) \\ &= \mathcal{D}(u)\,\sigma(v) + u\mathcal{D}\left(x_i^{\varepsilon}\right)\sigma\left(x_i^{-\varepsilon}v\right) - ux_i^{\varepsilon}\mathcal{D}\left(x_i^{\varepsilon}\right)\sigma\left(x_i^{-\varepsilon}\right)\sigma(v) \\ &\quad + u\mathcal{D}(v) \\ &= \mathcal{D}(u)\,\sigma(v) + u\mathcal{D}(v), \quad \varepsilon_i = \pm 1\end{aligned}$$

shows that $\mathcal{D}$ is well defined on F_n. □

Proposition 4.4 *If F_n is a free group generated by $x_1, x_2, \ldots, x_n$ and w is a word in F_n, there are the following properties satisfied by the partial derivatives $\frac{\partial}{\partial x_i} : \mathbb{Z}F_n \to \mathbb{Z}F_n$, $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$ of the delta derivative.*

1. $\dfrac{\partial(w_1w_2)}{\partial x_i} = \dfrac{\partial w_1}{\partial x_i}\sigma(w_2) + w_1\dfrac{\partial w_2}{\partial x_i}$.
2. $\dfrac{\partial w^{-1}}{\partial x_i} = -w^{-1}\sigma(w)^{-1}\dfrac{\partial w}{\partial x_i}$.

3. $\dfrac{\partial x^{-n}}{\partial x} = \sigma\left(x^{n-1}\right) + \sigma\left(x^{n-2}\right)x + \cdots + \sigma(x)\,x^{n-2} + x^{n-1}$.

4. $\dfrac{\partial x^{-n}}{\partial x} = -\Big(\sigma\left(x^{-n}\right)x^{-1} + \sigma\left(x^{-(n-1)}\right)x^{-2} + \cdots + \sigma\left(x^{-2}\right)x^{-(n-1)}$
$+ \sigma\left(x^{-1}\right)x^{-n}\Big)$, *for* $n \geq 1$.

Proof Property 1 is a repetition of the product rule and the other properties are the same as the properties of Lemma 4.2. □

5. Delta polynomial

Let $G = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n\rangle$ be a finitely presented group. Regarding the relations $r_1, r_2, \ldots, r_n$ as words in the x_j's, we form the Jacobian matrix $J = \frac{\partial x_i}{\partial x_j}$ of partial delta derivatives where these derivatives can be simplified by using relations in G. Denote by $J^\phi = \left(\frac{\partial x_i}{\partial x_j}\right)^\phi$ the image of the Jacobian under the abelianization map. The matrix J^ϕ is called the ϕ-Jacobian matrix or delta matrix of G.

Definition 5.1 *We consider the* ϕ*-delta matrix* J^ϕ *for a knot group* $\pi_1(S^3 \setminus K)$ *with respect to a presentation* $G = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n\rangle$*. Let* E *be the ideal generated by the* $(n-1)\times(n-1)$ *minors of* J^ϕ*. The delta polynomial* $\mathcal{D}_K(t)$ *is a GCD of* E *up to multiplying by any power* $\pm t^k \sigma(t)^l$*,* $k, l \in \mathbb{Z}$*.*

If $\mathcal{D}_{K_1}(t)$ and $\mathcal{D}_{K_2}(t)$ are polynomials that are equal up to such a factor, we write $\mathcal{D}_{K_1}(t) \doteq \mathcal{D}_{K_2}(t)$. If $G = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n\rangle$ is a Wirtinger presentation of a knot group, see [15], then any one of the $(n-1)\times(n-1)$ minors of J^ϕ can be taken to be $\mathcal{D}_K(t)$.

Since the delta derivative is a general case of the free derivative, the delta polynomial is also a general case of the Alexander polynomial in such a manner that if we take $\sigma(t) = 1$ in the delta polynomial then we have the Alexander polynomial. Thus, according to Section 3, the delta derivative is the Alexander polynomial for $\sigma(t) = 1$ and the difference polynomial for $\sigma(t) = t + 1$.

Example 5.2 *Let* K *denote the trefoil. A Wirtinger presentation of the knot group of* K *is given in [6] as follows:*

$$G = \left\langle x, y, z | r_1 = xyz^{-1}y^{-1},\ r_2 = yzx^{-1}z^{-1},\ r_3 = zxy^{-1}x^{-1}\right\rangle,$$

where the defining relation r_i *is a relation obtained in the crossing* c_i *of the diagram of the trefoil. Then*

$$\frac{\partial r_1}{\partial x} = \sigma\left(yz^{-1}y^{-1}\right), \quad \frac{\partial r_1}{\partial y} = x\sigma\left(z^{-1}y^{-1}\right) - xyz^{-1}y^{-1}\sigma\left(y^{-1}\right),$$
$$\frac{\partial r_1}{\partial z} = -xyz^{-1}\sigma\left(z^{-1}\right)\sigma\left(y^{-1}\right).$$

$$\frac{\partial r_2}{\partial x} = -yzx^{-1}\sigma\left(x^{-1}\right)\sigma\left(z^{-1}\right), \quad \frac{\partial r_2}{\partial y} = \sigma\left(zx^{-1}z^{-1}\right),$$
$$\frac{\partial r_2}{\partial z} = y\sigma\left(x^{-1}z^{-1}\right) - yzx^{-1}z^{-1}\sigma\left(z^{-1}\right).$$

$$\frac{\partial r_3}{\partial x} = z\sigma\left(x^{-1}y^{-1}\right) - zxy^{-1}x^{-1}\sigma\left(x^{-1}\right), \quad \frac{\partial r_3}{\partial y} = -zxy^{-1}\sigma\left(y^{-1}\right)\sigma\left(x^{-1}\right),$$

$$\frac{\partial r_3}{\partial z} = xy^{-1}x^{-1}.$$

We obtain the ϕ-delta matrix of them under the abelianization map

$$J^{\phi} = \begin{bmatrix} \sigma(t)^{-1} & t\sigma(t)^{-2} - \sigma(t)^{-1} & -t\sigma(t)^{-2} \\ -t\sigma(t)^{-2} & \sigma(t)^{-1} & t\sigma(t)^{-2} - t\sigma(t)^{-1} \\ t\sigma(t)^{-2} - \sigma(t)^{-1} & -t\sigma(t)^{-2} & \sigma(t)^{-1} \end{bmatrix}.$$

Since $|J^{\phi}| = 0$, the 2×2 minor

$$M_{11} = \begin{bmatrix} \sigma(t)^{-1} & t\sigma(t)^{-2} - \sigma(t)^{-1} \\ -t\sigma(t)^{-2} & \sigma(t)^{-1} \end{bmatrix},$$

for instance, is a presentation matrix and

$$|M_{11}| = \sigma(t)^{-2} + t^2\sigma(t)^{-4} - t\sigma(t)^{-3}.$$

Hence the delta polynomial of K is

$$\mathcal{D}_K(t) = t^2\sigma(t)^{-2} - t\sigma(t)^{-1} + 1$$

up to multiplying by $\sigma(t)^{-2}$. Then the Alexander polynomial of K is

$$\nabla_K(t) = t^2 - t + 1$$

and the difference polynomial of K,

$$\Delta_K(t) = t^2 + t + 1$$

up to multiplying by $\frac{1}{t^2+1}$.

Theorem 5.3 *If K is a knot or link, then the delta polynomial, $\mathcal{D}_K(t)$, of the knot K is an invariant of ambient isotopy.*

Proof In order to prove that the delta polynomial is an invariant of ambient isotopy, we must investigate the behavior of $\mathcal{D}_K(t)$ under the Reidemeister moves given in Figure 1. Here we shall investigate the behavior of $\mathcal{D}_K(t)$ under the diagrams given in Figure 2.

Let K be a knot with n crossings. For example, let the generators that are meted by the crossings c_{n-1} and c_n of the knot K be given as in Figure 2. $G = \langle x_1, x_2, \dots, x_n | r_1, r_2, \dots, r_n \rangle$ is a Wirtinger presentation of the group of the diagram K in Figure 2. We can obtain the defining relations $r_{n-1} = x_{n-2}x_nx_{n-3}^{-1}x_n^{-1}$ at the crossing c_{n-1} and $r_n = x_{n-1}x_{n-2}x_n^{-1}x_{n-2}^{-1}$ at the crossings c_n. Then

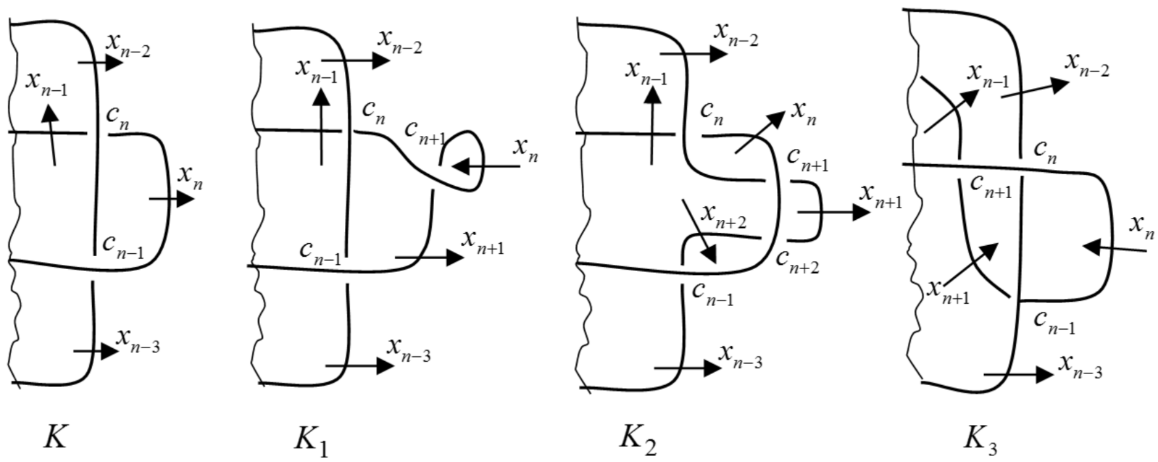

Figure 2. Diagrams for the proof of Theorem 5.3. For the first Reidemeister move: $K \leftrightarrow K_1$; for the second Reidemeister move: $K \leftrightarrow K_2$; for the third Reidemeister move: $K \leftrightarrow K_3$.

$$\frac{\partial r_n}{\partial x_n} = -x_{n-1}x_{n-2}x_n^{-1}\sigma(x_n^{-1})\sigma(x_{n-2}^{-1}),$$
$$\frac{\partial r_n}{\partial x_{n-1}} = \sigma(x_{n-2}x_n^{-1}x_{n-2}^{-1}),$$
$$\frac{\partial r_n}{\partial x_{n-2}} = x_{n-1}\sigma(x_n^{-1}x_{n-2}^{-1}) - x_{n-1}x_{n-2}x_n^{-1}x_{n-2}^{-1}\sigma(x_{n-2}^{-1}),$$
$$\frac{\partial r_{n-1}}{\partial x_n} = x_{n-2}\sigma(x_{n-3}^{-1}x_n^{-1}) - x_{n-2}x_nx_{n-3}^{-1}x_n^{-1}\sigma(x_n^{-1}),$$
$$\frac{\partial r_{n-1}}{\partial x_{n-2}} = \sigma(x_nx_{n-3}^{-1}x_n^{-1}),$$
$$\frac{\partial r_{n-1}}{\partial x_{n-3}} = -x_{n-2}x_nx_{n-3}^{-1}\sigma(x_{n-3}^{-1})\sigma(x_n^{-1}).$$

By the abelianization map,

$$(\frac{\partial r_n}{\partial x_n})^{\Phi} = -t\sigma(t)^{-2}, \quad (\frac{\partial r_n}{\partial x_{n-1}})^{\Phi} = \sigma(t)^{-1}, \quad (\frac{\partial r_n}{\partial x_{n-2}})^{\Phi} = t\sigma(t)^{-2} - \sigma(t)^{-1},$$
$$(\frac{\partial r_{n-1}}{\partial x_n})^{\Phi} = t\sigma(t)^{-2} - \sigma(t)^{-1}, \quad (\frac{\partial r_{n-1}}{\partial x_{n-2}})^{\Phi} = \sigma(t)^{-1}, \quad (\frac{\partial r_{n-1}}{\partial x_{n-3}})^{\Phi} = -t\sigma(t)^{-2}.$$

For simplicity, we write $a = \sigma(t)^{-1}$, $b = \sigma(t)^{-2}$. Hence we obtain the $n \times n$ Jacobian matrix J^{ϕ} of derivatives

$$J^{\phi} = \begin{bmatrix} (\frac{\partial r_1}{\partial x_1})^{\Phi} & \cdots & (\frac{\partial r_1}{\partial x_{n-3}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-2}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-1}})^{\Phi} & (\frac{\partial r_1}{\partial x_n})^{\Phi} \\ \cdots & \vdots & & & & \\ (\frac{\partial r_{n-1}}{\partial x_{n-1}})^{\Phi} & \cdots & -b & a & 0 & b-a \\ (\frac{\partial r_n}{\partial x_n})^{\Phi} & & 0 & b-a & a & -b \end{bmatrix}.$$

Since $|J^{\phi}| = 0$, any one of the $(n-1)\times(n-1)$ minors of J^{ϕ}, for instance, M_{11} is a presentation matrix and $|M_{11}| = \mathcal{D}_K(t)$ up to multiplying by $\pm a^k b^l$, $k,l \in \mathbb{Z}$.

- *The behavior of $\mathcal{D}_K(t)$ under the first Reidemeister move.*

  Since the diagram K is equivalent to K_1 in Figure 2 under the first Reidemeister move, we must examine the behavior of $\mathcal{D}_{K_1}(t)$ under the first Reidemeister move.

  Let $G_1 = \langle x_1, x_2, \ldots, x_n, x_{n+1} | r_1, r_2, \ldots, r_n, r_{n+1}\rangle$ be a Wirtinger presentation of the group of the diagram K_1. We can obtain the defining relations $r_{n-1} = x_{n-2}x_n x_{n-3}^{-1}x_n^{-1}$ at the crossing c_{n-1}, $r_n = x_{n-1}x_{n-2}x_{n+1}^{-1}x_{n-2}^{-1}$ at the crossings c_n, and $r_{n+1} = x_{n+1}x_n^{-1}$ at the crossings c_{n+1}. Hence, by the abelianization map, we obtain the following $(n+1)\times(n+1)$ Jacobian matrix J_1^{ϕ} of derivatives:

$$J_1^{\phi} = \begin{bmatrix} (\frac{\partial r_1}{\partial x_1})^{\Phi} & \cdots & (\frac{\partial r_1}{\partial x_{n-3}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-2}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-1}})^{\Phi} & (\frac{\partial r_1}{\partial x_n})^{\Phi} & 0 \\ \cdots & \vdots & & & & & \\ (\frac{\partial r_{n-1}}{\partial x_{n-1}})^{\Phi} & \cdots & -b & a & 0 & b-a & 0 \\ (\frac{\partial r_n}{\partial x_n})^{\Phi} & & 0 & b-a & a & 0 & -b \\ 0 & & 0 & 0 & 0 & -a & a \end{bmatrix}$$

$$= \begin{bmatrix} (\frac{\partial r_1}{\partial x_1})^{\Phi} & \cdots & (\frac{\partial r_1}{\partial x_{n-3}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-2}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-1}})^{\Phi} & (\frac{\partial r_1}{\partial x_n})^{\Phi} & 0 \\ \cdots & \vdots & & & & & \\ (\frac{\partial r_{n-1}}{\partial x_{n-1}})^{\Phi} & \cdots & -b & a & 0 & b-a & 0 \\ (\frac{\partial r_n}{\partial x_n})^{\Phi} & & 0 & b-a & a & -b & -b \\ 0 & & 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

  Since $|J_1^{\phi}| = a|J^{\phi}| = 0$, the $(n-1)\times(n-1)$ minors of J_1^{ϕ} are equal to the corresponding $(n-1)\times(n-1)$ minors of J^{ϕ} and thus $\mathcal{D}_K(t) \doteq \mathcal{D}_{K_1}(t)$. In that case $\mathcal{D}_K(t)$ is unchanged under the first Reidemeister move.

- *The behavior of $\mathcal{D}_K(t)$ under the second Reidemeister move.*

  Since the diagram K is equivalent to K_2 in Figure 2 under the second Reidemeister move, to see that $\mathcal{D}_K(t)$ is unchanged under the second Reidemeister move we must prove that $\mathcal{D}_K(t) \doteq \mathcal{D}_{K_2}(t)$. For this we must examine the behaviour of $\mathcal{D}_{K_2}(t)$ under the second Reidemeister move.

  Let $G_2 = \langle x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2} | r_1, r_2, \ldots, r_n, r_{n+1}, r_{n+2}\rangle$ be a Wirtinger presentation of the group of the diagram K_2. We have the defining relations $r_{n-1} = x_{n+2}x_n x_{n-3}^{-1}x_n^{-1}$ at the crossing c_{n-1}, $r_n = x_{n-1}x_{n-2}x_n^{-1}x_{n-2}^{-1}$ at the crossings c_n, $r_{n+1} = x_{n-2}x_n x_{n+1}^{-1}x_n^{-1}$ at the crossings c_{n+1}, $r_{n+2} = x_n x_{n+1}x_n^{-1}x_{n+2}^{-1}$ at the crossings c_{n+2}. Hence, by the abelianization map, we obtain the following $(n+2)\times(n+2)$ Jacobian matrix J_2^{ϕ} of derivatives:

$$J_2^{\phi} = \begin{bmatrix} (\frac{\partial r_1}{\partial x_1})^{\Phi} & \cdots & (\frac{\partial r_1}{\partial x_{n-3}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-2}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-1}})^{\Phi} & (\frac{\partial r_1}{\partial x_n})^{\Phi} & 0 & 0 \\ \cdots & \vdots & & & & & & \\ (\frac{\partial r_{n-1}}{\partial x_{n-1}})^{\Phi} & \cdots & -b & 0 & 0 & b-a & 0 & a \\ (\frac{\partial r_n}{\partial x_n})^{\Phi} & & 0 & b-a & a & -b & 0 & 0 \\ 0 & & 0 & a & 0 & b-a & -b & 0 \\ 0 & & 0 & 0 & 0 & a-b & b & -a \end{bmatrix}$$

$$= \begin{bmatrix} (\frac{\partial r_1}{\partial x_1})^{\Phi} & \cdots & (\frac{\partial r_1}{\partial x_{n-3}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-2}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-1}})^{\Phi} & (\frac{\partial r_1}{\partial x_n})^{\Phi} & 0 & 0 \\ \cdots & \vdots & & & & & & \\ (\frac{\partial r_{n-1}}{\partial x_{n-1}})^{\Phi} & \cdots & -b & a & 0 & b-a & 0 & 0 \\ (\frac{\partial r_n}{\partial x_n})^{\Phi} & & 0 & b-a & a & -b & 0 & 0 \\ 0 & & 0 & a & a & -a & -b & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 & -a \end{bmatrix}.$$

Since $|J_2^{\phi}| = -ab|J^{\phi}| = 0$, the $(n-1)\times(n-1)$ minors of J_2^{ϕ} are equal to the corresponding $(n-1)\times(n-1)$ minors of J^{ϕ} and thus $\mathcal{D}_K(t) \doteq \mathcal{D}_{K_2}(t)$. Thus $\mathcal{D}_K(t)$ is unchanged under the second Reidemeister move.

- *The behavior of $\mathcal{D}_K(t)$ under the third Reidemeister move.*

In order to show that $\mathcal{D}_K(t)$ is unchanged under the third Reidemeister move, it is sufficient to prove that $\mathcal{D}_K(t) \doteq \mathcal{D}_{K_3}(t)$ for the diagrams K and K_3 in Figure 2.

It is easy to see that, in the presence of the first and the second Reidemeister moves, the diagram K_3 is equivalent to the third Reidemeister move as seen in Figure 3.

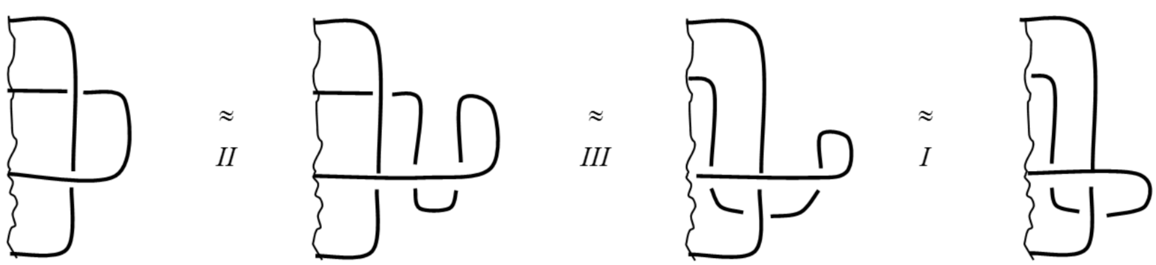

Figure 3. The schematic proof of the equivalence of the diagram K to the diagram K_3.

Let $G_3 = \langle x_1, x_2, \ldots, x_n, x_{n+1} | r_1, r_2, \ldots, r_n, r_{n+1}\rangle$ be a Wirtinger presentation of the group of the diagram K_3. Then we can write the relations $r_{n-1} = x_{n-2}x_nx_{n-3}^{-1}x_n^{-1}$ at the crossing c_{n-1}, $r_n = x_{n-1}x_nx_{n+1}^{-1}x_n^{-1}$

at the crossings c_n, $r_{n+1} = x_{n+1}x_{n-3}x_n^{-1}x_{n-3}^{-1}$ at the crossings c_{n+1}. By the abelianization map, we obtain the following $(n+1)\times(n+1)$ Jacobian matrix J_3^{ϕ} of derivatives:

$$J_3^{\phi} = \begin{bmatrix} (\frac{\partial r_1}{\partial x_1})^{\Phi} & \cdots & (\frac{\partial r_1}{\partial x_{n-3}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-2}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-1}})^{\Phi} & (\frac{\partial r_1}{\partial x_n})^{\Phi} & 0 \\ \cdots & \vdots & & & & & \\ (\frac{\partial r_{n-1}}{\partial x_{n-1}})^{\Phi} & \cdots & -b & a & 0 & b-a & 0 \\ (\frac{\partial r_n}{\partial x_n})^{\Phi} & & 0 & 0 & a & b-a & -b \\ 0 & & b-a & 0 & 0 & -b & a \end{bmatrix}$$

$$= \begin{bmatrix} (\frac{\partial r_1}{\partial x_1})^{\Phi} & \cdots & (\frac{\partial r_1}{\partial x_{n-3}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-2}})^{\Phi} & (\frac{\partial r_1}{\partial x_{n-1}})^{\Phi} & (\frac{\partial r_1}{\partial x_n})^{\Phi} & 0 \\ \cdots & \vdots & & & & & \\ (\frac{\partial r_{n-1}}{\partial x_{n-1}})^{\Phi} & \cdots & -b & a & 0 & b-a & 0 \\ (\frac{\partial r_n}{\partial x_n})^{\Phi} & & 0 & b-a & a & -b & a-b \\ 0 & & 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

Since $|J_3^{\phi}| = a|J^{\phi}| = 0$, the $(n-1)\times(n-1)$ minors of J_3^{ϕ} are equal to the corresponding $(n-1)\times(n-1)$ minors of J^{ϕ} and $\mathcal{D}_K(t) \doteq \mathcal{D}_{K_3}(t)$. Thus proof is completed.

$\square$

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