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### Research Article

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# Introduction to disoriented knot theory

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**Abstract:** This paper is an introduction to disoriented knot theory, which is a generalization of the oriented knot and link diagrams and an exposition of new ideas and constructions, including the basic definitions and concepts such as disoriented knot, disoriented crossing and Reidemeister moves for disoriented diagrams, numerical invariants such as the linking number and the complete writhe, the polynomial invariants such as the bracket polynomial, the Jones polynomial for the disoriented knots and links.

**Keywords:** Disoriented diagrams, Disoriented crossing, Linking Number, Complete writhe, Bracket polynomial, Jones polynomial

**MSC:** 57M25, 57M27

## 1 Introduction

This paper gives an introduction to the subject of disoriented knot theory. We introduce the notion of a disoriented crossing and explain how to adapt the fundamental concepts and invariants of the knot theory to a setting in which we have disoriented crossing.

The disoriented knot and link diagrams arise when calculating the Jones [1, 2], HOMFLY [3] polynomials etc., using the oriented diagram structure of the state summation for the oriented knot and link diagrams. When we split a crossing of an oriented knot diagram using Kauffman's bracket model [4, 5], one of the occurring diagram is a disoriented diagram. Since the disoriented diagrams acquire orientations outside the category of knot and link diagrams, only unoriented and oriented knot and link diagrams have previously been considered in the literature for the development of knot theory.

In this paper, we put disoriented knot and link diagrams on the same footing as oriented knot and link diagrams. We define the concepts of a disoriented diagram and a disoriented crossing and give the Reidemeister moves for disoriented diagrams. We see that most classic basic concepts of knot theory are not invariant for the disoriented diagrams of a knot or link. Here, we redefine the linking number for the disoriented link diagrams and prove that the linking number is a link invariant of the link. We define a new concept called complete writhe instead of the classic writhe that is not a regular isotopy invariant for disoriented link diagrams. Thus, the normalized bracket polynomial (the Jones polynomial) by the complete writhe for disoriented link diagrams will be an invariant of the link.

The paper is organized as follows. Section 2 gives the definitions of the disoriented knot and link diagrams and disoriented crossing. In this section, we also give the brief description about of disoriented diagrams and the Reidemeister moves on the disoriented diagrams.

Sections 3 and 4 contain some numerical invariants. In Section 3, we define the linking number for the disoriented link diagrams, prove that the linking number is a link invariant and give two examples. In Section 4, we adapt the writhe for the disoriented knot and link diagrams. We define a new concept called complete

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writhe for the disoriented knot and link diagrams and prove that the complete writhe is invariant under the Reidemeister moves  $RII$  and  $RIII$  for the disoriented diagrams.

Section 5 presents a review of the bracket polynomial [4–6] and the Jones polynomial for the disoriented knot and link diagrams. We expand the bracket polynomial for disoriented diagrams of a knot (or link). So, the normalized bracket polynomial can also be extended for the disoriented knot and link diagrams. Since the normalized bracket polynomial (the Jones polynomial) is invariant under the first Reidemeister move for oriented knot diagrams, it is a stronger invariant than regular isotopy invariant for the disoriented diagrams. However, the Jones polynomial is not invariant under all the moves on disoriented diagrams. Therefore, it is not an invariant for the disoriented knot and link diagrams. In this section, we redefine the Jones polynomial for the disoriented knot and link diagrams by using complete writhe. This polynomial called the complete Jones polynomial is invariant under all the moves on disoriented diagrams. Thus it is a knot and link invariant. Moreover, the complete Jones polynomial of each disoriented diagram of a knot is equal to the original Jones polynomial of the knot. The last part of this section contains a few examples.

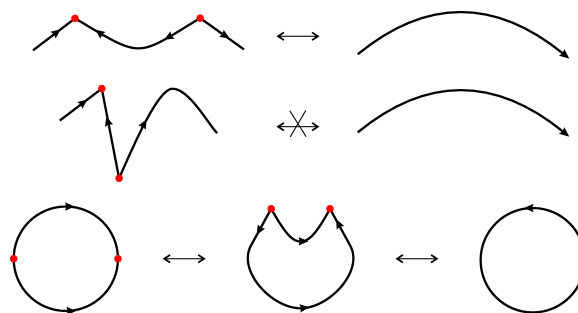
## 2 Defining disoriented knots and links

A knot is an embedding of a circle in three dimensional space  $\mathbb{R}^3$  (or  $S^3$ ). An oriented knot is a knot diagram which has been given an orientation. Taking into account the facts about the embedding, we can also define an oriented knot as an embedding of an oriented circle in three dimensional space. In a similar way, we define a disoriented knot.

**Definition 2.1.** *A disoriented circle is a circle upon which we have chosen two points, and have chosen an orientation of both of the arcs between those two points. We allow that the orientation of one of the arcs is the reverse of the orientation of the other.*

A few simple disoriented diagrams, a disoriented circle and their replacements have been illustrated in Fig. 1. The basic reduction move in Fig. 1 corresponds to elimination of two consecutive cusps on a simple loop. Note that it is allowed to the cancelation of consecutive cusps along a loop where the cusps both points to the same local region of the two local regions delineated by the loop but not allowed to the cancelation of a "zig-zag" where two consecutive cusps point to opposite local sides of the loop. A zig-zag is represented an oriented or disoriented virtual crossing of the oriented or disoriented diagram of a knot and link. Since we work on the classical knot and link diagrams in this paper, we do not encounter a zig-zag. Thus, for we can easily draw a disoriented knot or link diagram, we use the local disoriented curve of the form  $\rightarrow \cdot \leftarrow \cdot \rightarrow$  instead of the cusp with two points of the form  $\wedge \smile$ . Due to our present disclosure, a disoriented curve can be replaced with an oriented curve. Likewise, a disoriented circle can be replaced with an oriented circle. For detailed information about disoriented configurations, replacements and disoriented relations can be seen in [7–11].

Fig. 1. Simple Disoriented Diagrams and Replacements.



**Definition 2.2.** A disoriented knot is an embedding of a disoriented circle in three dimensional space  $\mathbb{R}^3$  (or  $S^3$ ). A disoriented link of  $k$ -components is an embedding of a disjoint union of  $k$  circles in  $\mathbb{R}^3$ , where at least one of circles is disoriented. Two disoriented knots equivalent if there is a continuous deformation of  $\mathbb{R}^3$  taking one to the other.

If  $K$  a disoriented knot (or link) in  $\mathbb{R}^3$ , its projection is  $\pi(K) \subset \mathbb{R}^2$ , where  $\pi$  is the projection along the  $z$ -axis onto the  $xy$ -plane. The projection is said to be regular projection if the preimage of a point of  $\pi(K)$  consists of either one or two points of  $K$ . If  $K$  has a regular projection, then we can define the corresponding disoriented diagram  $D$  by redrawing it with an arc near crossing (the place with two preimages in  $K$ ) to incorporate the overpass/underpass information.

**Definition 2.3.** A crossing of a disoriented knot  $K$  is disoriented if the overpass and underpass arcs of the crossing have opposite orientations. In other words, if  $A_1$  and  $A_2$  are the arcs of the disoriented circle of which  $K$  is an embedding, then one of the underpass and overpass arcs is  $A_1$ , and the other is  $A_2$ . If a crossing of a disoriented knot  $K$  is not disoriented, we say that it is oriented. An oriented knot is a disoriented knot with zero disoriented crossing. For example, see Fig. 3.

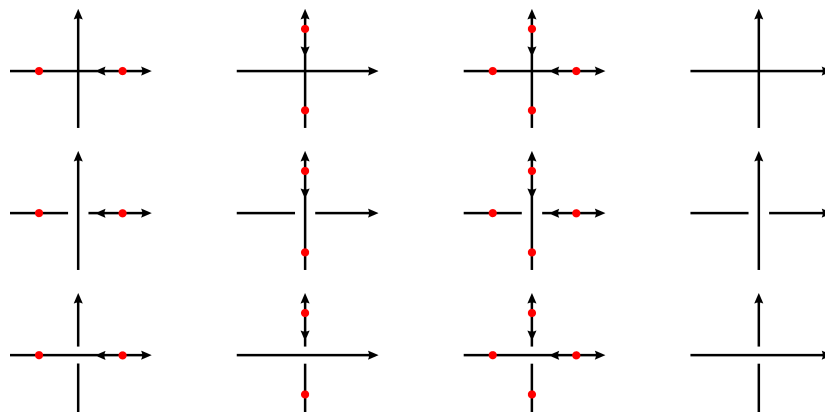
**Definition 2.4.** Let  $L$  be a link with exactly two components,  $K_1$  and  $K_2$ . Choose a disorientation of both  $K_1$  and  $K_2$ . Denote the two arcs of  $K_1$  by  $A_1^1$  and  $A_1^2$ , and denote the two arcs of  $K_2$  by  $A_2^1$  and  $A_2^2$ . We say that a crossing of  $L$  is disoriented if either of the following holds:

1. One of the underpass and overpass arcs of the crossings is  $A_1^1$ , and the other is  $A_2^1$  or  $A_2^2$ .
  2. One of the underpass and overpass arcs of the crossings is  $A_1^2$ , and the other is  $A_2^2$  or  $A_2^1$ .
- Otherwise, we say that the crossing is oriented.

**Remark 2.5.** Similar considerations apply to disoriented links with more than two components. We have not discussed here to avoid going beyond the purpose of the paper.

Although the underpass and overpass of an oriented crossing are the same with the underpass and overpass of a disoriented crossing, the signs of these crossings are not the same. Then, a disoriented crossing can not be replaced with an oriented crossing. We illustrate oriented and disoriented crossings in Fig. 2.

Fig. 2. Disoriented and Oriented Crossings.



The Reidemeister moves on disoriented diagrams generalize the Reidemeister moves for the oriented knot and link diagrams. We illustrate the Reidemeister moves on disoriented diagrams in Fig. 4, 5 and 6. The Reidemeister moves of types *II* and *III* for oriented diagrams are expanded on disoriented diagrams. A new disoriented curled move is added to Reidemeister move of type *I* for disoriented diagrams. Disoriented knot and link diagrams that can be connected by a finite sequence of these moves and their inverse moves are said

to be equivalent. We list the equivalence of these Reidemesiter moves illustrated in Fig. 4, 5 and 6 as below:

Planar moves :  $I'_0 \leftrightarrow I_0, I''_0 \leftrightarrow I_0$

The move  $RI : I \leftrightarrow I_0$

The move  $RI' : I' \leftrightarrow I_1$

The moves  $RII : \text{For } i \in \{0, 1, 2\}, J_i \leftrightarrow K_0, J'_i \leftrightarrow K_0,$

$L_i \leftrightarrow K'_0, L'_i \leftrightarrow K'_0$

The moves  $RIII : \text{For } i \in \{0, 1, 2\}, T_i \leftrightarrow S_i, T'_i \leftrightarrow S'_i, T''_i \leftrightarrow S''_i.$

Fig. 3. Disoriented Diagrams of the right-hand trefoil.

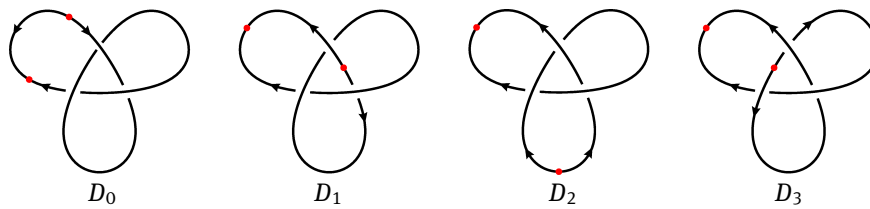


Fig. 4. Planar and First Reidemeister Moves for Disoriented Diagrams.

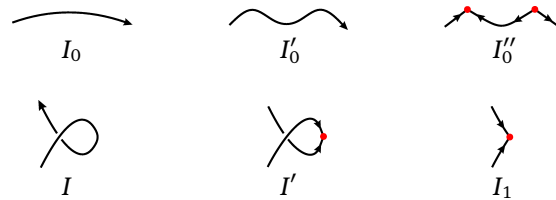


Fig. 5. Second Reidemeister Moves for Disoriented Diagrams.

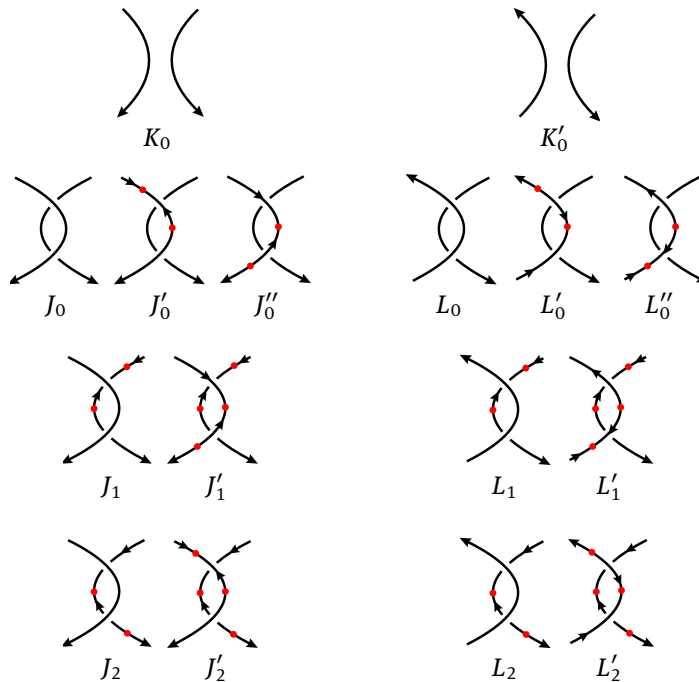
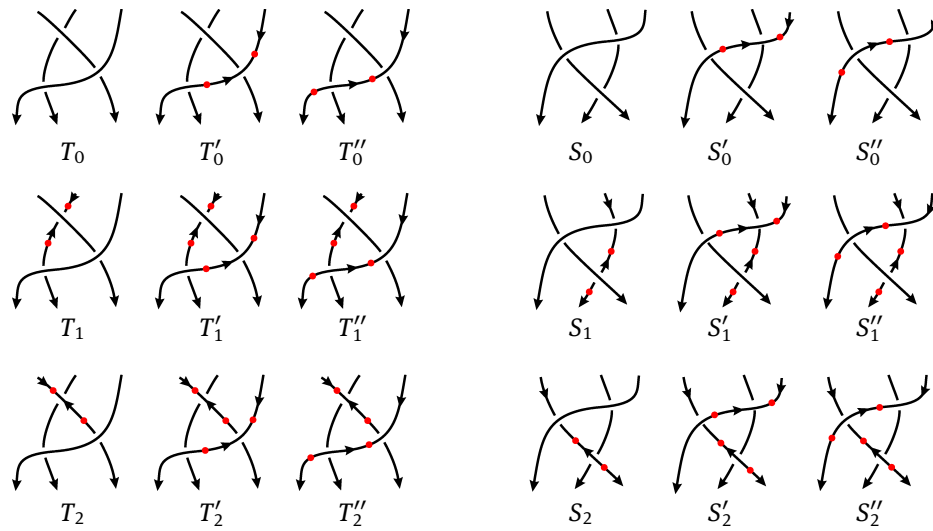


Fig. 6. Third Reidemeister Moves for Disoriented Diagrams.



**Remark 2.6.** Note that there are many other possible choices of the orientation of the arcs in Fig. 6.

We call regular isotopy to the equivalence relation generated by the moves *RII* and *RIII* (and the planar moves), ambient isotopy to the equivalence relation generated by the moves *RI*, *RII* and *RIII*, and complete ambient isotopy to the equivalence relation on disoriented diagrams that is generated by all the moves in Fig. 4, 5 and 6. Since there is no disoriented crossing of an oriented diagram, the complete ambient isotopy is equivalent to the ambient isotopy on the oriented diagrams. But, the complete ambient isotopy is more powerful than the ambient isotopy for the disoriented knot and link diagrams.

### 3 Linking number

In this section, we define the linking number for disoriented links and prove that the linking number of a disoriented link is its invariant.

**Definition 3.1.** Let  $L$  be a disoriented link with two components  $K_1$  and  $K_2$ . The linking number  $lk(L)$  is defined by formula

$$lk(L) = \frac{1}{2} \left| \sum_{o \in K_1 \sqcap K_2} \varepsilon(o) - \sum_{d \in K_1 \sqcap K_2} \varepsilon(d) \right|$$

where  $K_1 \sqcap K_2$  denotes the set of crossings of  $K_1$  with  $K_2$  (no self-crossings), where the first sum runs over the oriented crossings of  $K_1 \sqcap K_2$ , the second sum runs over the disoriented crossings of  $K_1 \sqcap K_2$ , and  $\varepsilon(o)$  and  $\varepsilon(d)$  denote the sign of an oriented crossing and the sign of a disoriented crossing of belonging to  $K_1 \sqcap K_2$ , respectively.

The following theorem gives that the linking numbers of all the disoriented diagrams of a link are equal.

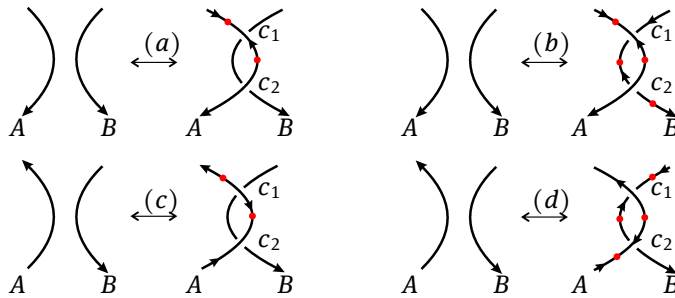
**Theorem 3.2.** The linking number  $lk(L)$  is an invariant of the link  $L$ .

*Proof.* Let  $D$  be a disoriented regular diagram of the link  $L$  with two components. We suppose that  $D'$  is another regular diagram of  $L$ . From our discussions so far, we know that we may obtain  $D'$  by performing, if necessary several times, the Reidemeister moves in Fig. 4, 5 and 6. Therefore, in order to prove the theorem, it is sufficient to show that the value of the linking number remains unchanged after each of the Reidemeister moves is performed on  $D$ .

The move  $RI$ : At the crossings of  $D$  at which we intend to apply the move  $RI$ , every section (edge) of such a crossing belongs to the same component. Therefore, applying the move  $RI$  does not affect the calculation of the linking number. In the same way, the move  $RI'$  does also not affect the calculation of the linking number.

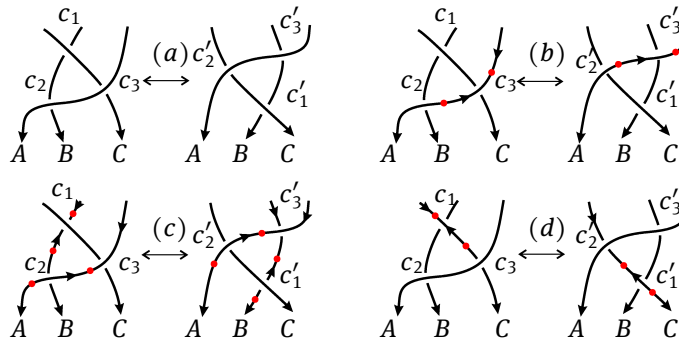
The moves  $RII$ : We shall only examine the effects of the cases  $J'_0, J'_2, L'_0$  and  $L'_1$  of the moves  $RII$  in Fig. 5 to the calculation of linking number. The remaining cases of the moves  $RII$  can be examined in a similar way. An application of the moves  $RII$  on  $D$  only has an effect on the linking number if  $A$  and  $B$  belong to different components, see Fig. 7. In the cases (a) and (c), the crossing  $c_1$  is disoriented and the crossing  $c_2$  is oriented. Also, the crossings  $c_1$  and  $c_2$  have the same signs. By Definition 3.1, we get  $\varepsilon(c_1) - \varepsilon(c_2) = 0$ . In the cases (b) and (d), both crossings are disoriented. Since the crossings  $c_1$  and  $c_2$  have opposite signs, we get  $\varepsilon(c_1) + \varepsilon(c_2) = 0$ . So, in each case, the linking number is unchanged under the second Reidemeister moves.

Fig. 7. Some Second Reidemeister Moves.



The moves  $RIII$ : Finally, let us consider the effect of the moves  $RIII$  on  $D$ . We only consider the effect of the cases  $T_0$  and  $S_0, T'_0$  and  $S'_0, T'_1$  and  $S'_1, T_2$  and  $S_2$  of the moves  $RIII$  in Fig. 6 to the calculation of linking number. That is, we only examine the effect on the signs of the crossings  $c_1, c_2, c_3$  and  $c'_1, c'_2, c'_3$  in Fig. 8.

Fig. 8. Some Disoriented Third Reidemeister Moves.



To be identical the effects of the cases  $T_i$  and  $S_i$ , the following equations should always be hold:

If all crossings are oriented or disoriented,

$$\varepsilon(c_1) = \varepsilon(c'_1), \quad \varepsilon(c_2) = \varepsilon(c'_2), \quad \varepsilon(c_3) = \varepsilon(c'_3).$$

If one of the crossings  $c_2$  and  $c'_2$  is oriented while other is disoriented or one of the crossings  $c_3$  and  $c'_3$  is oriented while other is disoriented

$$\varepsilon(c_1) = \varepsilon(c'_1), \quad \varepsilon(c_2) = -\varepsilon(c'_2), \quad \varepsilon(c_3) = \varepsilon(c'_3)$$

or

$$\varepsilon(c_1) = \varepsilon(c'_1), \quad \varepsilon(c_2) = \varepsilon(c'_2), \quad \varepsilon(c_3) = -\varepsilon(c'_3).$$

If one of the crossings  $c_2$  and  $c'_3$  is oriented while other is disoriented and one of the crossings  $c_3$  and  $c'_2$  is oriented while other is disoriented,

$$\varepsilon(c_1) = \varepsilon(c'_1), \quad \varepsilon(c_2) = -\varepsilon(c'_3), \quad \varepsilon(c_3) = -\varepsilon(c'_2).$$

If  $A$ ,  $B$  and  $C$  belong to the same component for the only case in Fig. 8, then the linking number is unaffected. So we suppose that  $A$  belong to a different component than  $B$  and  $C$ . Then the parts that have an effect on linking number is the sum of the signs of the crossings  $c_1$ ,  $c_2$  and  $c'_1$ ,  $c'_3$ . Since the crossings are oriented in the case (a), we have

$$\varepsilon(c_1) + \varepsilon(c_2) = \varepsilon(c'_1) + \varepsilon(c'_3)$$

by Definition 3.1. Since the crossing  $c'_3$  is disoriented and others are oriented in the case (b), we have

$$\varepsilon(c_1) + \varepsilon(c_2) = \varepsilon(c'_1) - \varepsilon(c'_3).$$

Since the crossing  $c'_3$  is oriented and others are disoriented in the case (c), we have

$$\varepsilon(c_1) + \varepsilon(c_2) = \varepsilon(c'_3) - \varepsilon(c'_1).$$

Since the crossing  $c_1$  and  $c'_1$  are disoriented and others are oriented in the case (d), we have

$$\varepsilon(c_2) - \varepsilon(c_1) = \varepsilon(c'_3) - \varepsilon(c'_1).$$

Thus, none of the sums does cause any change to the linking number. The other cases, (i.e. the various possibilities for the components that  $A$ ,  $B$  and  $C$  belong to) can be treated in a similar manner. The remaining cases of the moves *RIII* can be examined in a similar way. Hence, the linking number remains unchanged when we apply the moves *RIII*.  $\square$

We suppose now that  $L$  is a disoriented link with  $n$  components,  $K_1, K_2, \dots, K_n$ . With regard to two components,  $K_i$  and  $K_j$ ,  $i < j$ , we may define as an extension of the linking number  $lk(L) = lk(K_i, K_j)$ ,  $1 \leq i < j \leq n$ . This approach will give us  $n(n-1)/2$  linking numbers, and their sum,

$$lk(L) = \sum_{1 \leq i < j \leq n} lk(K_i, K_j)$$

is called the total linking number of  $K$ . One can show that, in fact, the total linking number of  $K$  is an invariant of  $K$ .

**Remark 3.3.** *The disoriented linking number is always equal to the oriented linking number, regardless of the choice of disorientation. Indeed, if one can change the sign of an oriented crossing by making disoriented, then it will be changed back again in the sum defining the linking number.*

**Example 3.4.** 1. *Let  $L$  be any disoriented diagram of unlink in Fig. 9. In the case (a), there are two oriented crossings which have the opposite sign. So,  $\varepsilon(o) = 0$ . In the cases (b) and (c), there are a disoriented crossing and an oriented crossing which have the same sign. So,  $\varepsilon(o) - \varepsilon(d) = 0$ . In the case (d), there are two disoriented crossings which have the opposite sign. So,  $\varepsilon(d) = 0$ . Thus, in each case, we get  $lk(L) = 0$ .*

2. *Let  $L$  be any disoriented diagram of link in Fig. 10. In the case (a), there are two oriented crossings which have the same sign. So,  $\varepsilon(o) = 2$ . In the cases (b) and (c), there are a disoriented crossing and an oriented crossing which have the opposite sign. So,  $|\varepsilon(o) - \varepsilon(d)| = 2$ . In the case (d), there are two disoriented crossings which have the negative sign. So,  $\varepsilon(d) = -2$  and  $|\varepsilon(o) - \varepsilon(d)| = 2$ . Thus, in each case, we get  $lk(L) = 1$ .*

Fig. 9. Some Disoriented Diagrams of Unlink

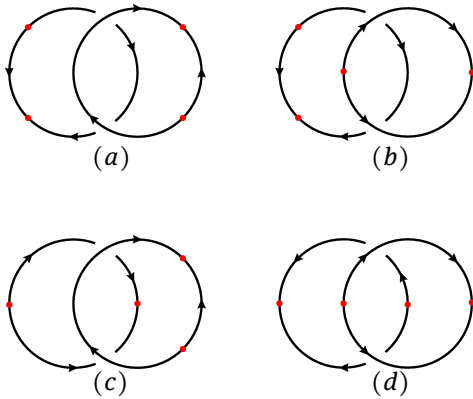
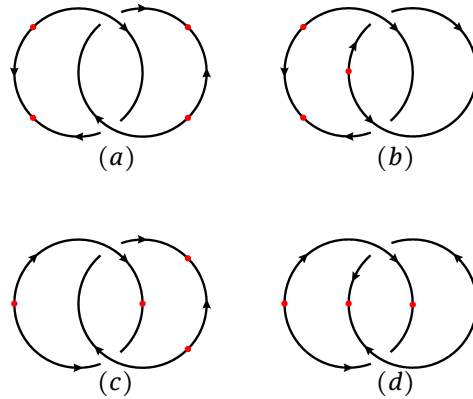


Fig. 10. Some Disoriented Diagrams of Hopf Link



## 4 Writhe

Recall that the classical writhe  $w(D)$  of a regular diagram  $D$  of a knot (or a link) is the sum of the signs of all the crossings of  $D$ . This definition of the writhe can be also adapted to disoriented diagram  $D$  of a knot. The classical writhe is not a knot invariant. However, regularly isotopic oriented knot diagrams have the same writhe. It can be easily understood that the classical writhe is not invariant under the moves  $RII$  and  $RIII$  for disoriented diagrams.

Now, we define a new writhe called complete writhe for a disoriented diagram of a knot or link and prove that the complete writhes of all the disoriented diagram of a knot are equal.

**Definition 4.1.** Let  $D$  be a disoriented regular diagram of a knot (or link)  $K$ . The complete writhe of  $D$ ,  $cw(D)$ , is defined by formula

$$cw(D) = \sum_o \varepsilon(o) - \sum_d \varepsilon(d),$$

where the first sum runs over the oriented crossings of  $D$ , the second sum runs over the disoriented crossings of  $D$  and  $\varepsilon(o)$  and  $\varepsilon(d)$  denote the sign of an oriented crossing and the sign of a disoriented crossing of belonging to  $D$ , respectively.

**Theorem 4.2.** The complete writhe  $cw(D)$  is a regular isotopy invariant of the disoriented diagram  $D$ .

*Proof.* If the proof of Theorem 3.2 is adopted for all crossings of the disoriented diagram  $D$ , it is easy to see that the effects of the moves  $RII$  and  $RIII$  on the linking number are identical to those on the complete writhe  $cw(D)$ . □

The complete writhe  $cw(D)$  of a disoriented diagram  $D$  is not invariant under the moves  $RI$  and  $RI'$ .

**Theorem 4.3.** The complete writhes of all the disoriented diagrams of a non-trivial knot (or non-trivial link) are same.

*Proof.* Let  $K$  be non-trivial knot (or non-trivial link) of  $n$  crossings. We denote an oriented diagram of the knot  $K$  by  $D_0$ . Let  $cw(D_0) = k$ ,  $k \in \mathbb{Z}$ . Let  $k_1$  be the sum of the positive signs of the crossings in the diagram  $D_0$  and  $k_2$  be the sum of the negative signs of the crossings in  $D_0$ . Then, we have

$$cw(D_0) = k = k_1 + k_2.$$

Now, let  $D_1$  denotes a disoriented diagram with one disoriented crossing. If the sign of the disoriented crossing is negative, by definition of complete writhe

$$cw(D_1) = (k_1 - 1) + k_2 + 1 = k.$$



If the sign of the disoriented crossing is positive,

$$cw(D_1) = k_1 + (k_2 + 1) - 1 = k.$$

Similarly, let  $D_m$  denotes a disoriented diagram of the knot  $K$ , where the sum of the signs of disoriented crossings of  $D_m$  is  $m \leq k$ . If  $m$  is positive,

$$cw(D_m) = k_1 + (k_2 + m) - m = k.$$

and if  $m$  is negative,

$$cw(D_m) = (k_1 - m) + k_2 + m = k.$$

Thus, for every disoriented diagram,  $D$  of the knot  $K$ , we have  $cw(D) = k, k \in \mathbb{Z}$ . □

**Remark 4.4.** *It can be understood from the proof of Theorem 4.3 that the complete writhe of any disorientation is equal to the classical writhe with respect to an orientation.*

As shown in the following example, all the disoriented diagrams of a knot have the same complete writhe, although each disoriented diagram of it has a different writhe.

**Example 4.5.** *For disoriented diagrams drawn in Fig. 3 of the right-hand trefoil, we have  $w(D_0) = 3, w(D_1) = 1, w(D_2) = -1, w(D_3) = -3$  and  $cw(D_i) = 3, i \in \{0, 1, 2, 3\}$ .*

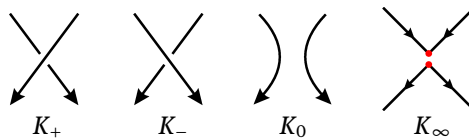
## 5 Polynomial invariants for disoriented knot diagrams

In this section, we give the bracket polynomial and the normalized bracket polynomial for disoriented knots and links. We show that the bracket polynomial is a regular isotopy invariant for disoriented knots and links. But, the normalized bracket polynomial is not an invariant for disoriented knots and links. For the normalized bracket polynomial to be an invariant of the disoriented knot we redefine the original normalized bracket polynomial by considering the notion of disoriented crossing that we call the complete normalized polynomial or the complete Jones polynomial. The construction of the complete normalized polynomial invariant begins with the disoriented summation of the bracket polynomial. This means that each local smoothing is either an oriented smoothing or a disoriented smoothing as illustrated in Fig. 11. The sufficient information about these smoothings can be found in Kauffman's works [7, 9]. The bracket expansion for the crossings with both positive and negative sign in an oriented knot and link diagram can be written as an oriented bracket state model:

$$\begin{aligned} \langle K_+ \rangle &= A \langle K_0 \rangle + A^{-1} \langle K_\infty \rangle \\ \langle K_- \rangle &= A^{-1} \langle K_0 \rangle + A \langle K_\infty \rangle \\ \delta &= -A^2 - A^{-2}, \quad \langle \bigcirc \sqcup D \rangle = \delta \langle D \rangle \end{aligned} \tag{1}$$

where  $K_+, K_-, K_0$  and  $K_\infty$  are diagrams in Fig. 11,  $\bigcirc$  and  $D$  is an oriented diagram with zero-crossing of unknot and an oriented knot or link diagram, respectively and  $\sqcup$  denotes disjoint union.

**Fig. 11.** Crossings and Smoothings.



We can use this model for both oriented and disoriented crossings in a disoriented knot and link diagram. We call the extended bracket polynomial which we have obtained polynomial from the model (1) for the

disoriented knot and link diagrams. As in the standard bracket polynomial, in corresponding disoriented state summation expansion of the extended bracket polynomial we have

$$\langle D \rangle = \sum_S \langle D|S \rangle \delta^{|S|-1}$$

where  $S$  runs over the oriented and the disoriented bracket states of the disoriented diagram  $D$ ,  $\langle D|S \rangle$  is the usual product of vertex weights and  $|S|$  is the number of the circle in the state  $S$ .

**Theorem 5.1.** *The extended bracket polynomial is a regular isotopy invariant for the disoriented knot and link diagrams.*

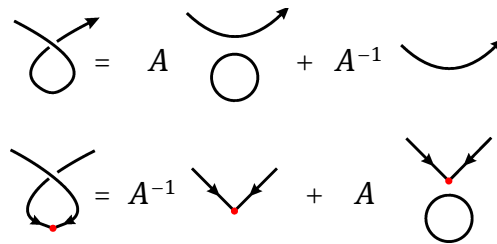
*Proof.* The proof is the same as oriented ones. The proof for oriented link diagrams given in [7]. □

The extended bracket polynomial is not an invariant of the moves  $RI$  and  $RI'$  for the disoriented knot and link diagrams. Its behavior under these moves is examined in the following lemma.

**Lemma 5.2.**  $\langle I \rangle = (-A^3)\langle I_0 \rangle$  and  $\langle I' \rangle = (-A^3)\langle I_1 \rangle$ , where  $I, I_0, I'$  and  $I_1$  are local diagrams given Fig. 4.

*Proof.* From Fig. 12, we obtain easily  $\langle I \rangle = (A\delta + A^{-1})\langle I_0 \rangle = (-A^3)\langle I_0 \rangle$  and  $\langle I' \rangle = (A^{-1} + A\delta)\langle I_1 \rangle = (-A^3)\langle I_1 \rangle$ . Note that  $I$  and  $I'$  have the opposite signs. □

Fig. 12. Oriented and Disoriented First Reidemeister Move.



The extended bracket polynomial is normalized to an invariant  $f_D(A)$  of all oriented and disoriented moves except the  $RI'$  move by the formula

$$f_D(A) = (-A^3)^{-w(K)} \langle D \rangle$$

where  $w(D)$  is the writhe of the disoriented diagram  $D$ . The polynomial  $f_D(A)$  is the extension of the normalized bracket polynomial by Kauffmann [4, 5] to disoriented knot and link diagrams. The Jones polynomial,  $V_D(t)$  is given in terms of this model by the formula

$$V_D(t) = f_D(t^{-1/4}).$$

This definition is a direct extension to the disoriented knot and link category of the state sum model for the original Jones polynomial. It is straightforward to verify the invariances stated above, see [4, 5]. In this way, we have the Jones polynomial for the disoriented knot and link diagrams. The original Jones polynomial is not an invariant for the disoriented links. Indeed, it is not invariant under the  $RI'$  move, see Example 5.7.

We now redefine the normalized bracket polynomial for the disoriented knot and link diagrams that we call the complete normalized polynomial and show that the complete normalized polynomial is invariant under all the moves of the disoriented knot and link diagrams.

**Definition 5.3.** *We define a polynomial  $\mathcal{T}_K \in \mathbb{Z}[A, A^{-1}]$  for a disoriented diagram  $D$  of a knot (or link)  $K$  by the formula*

$$\mathcal{T}_K(A) = (-A^3)^{-cw(D)} \langle D \rangle$$

where  $cw(D)$  is the complete writhe of disoriented diagram  $D$ . We call  $\mathcal{T}_K$  the complete normalized polynomial of the extended bracket polynomial by the complete writhe.

**Theorem 5.4.** *The complete normalize polynomial is a complete ambient isotopy invariant for the disoriented knot and link diagrams.*

*Proof.* Since  $cw(D)$  is a regular isotopy invariant, so  $(-A^3)^{-cw(D)}$ , and  $\langle D \rangle$  is also a regular isotopy invariant, it is follows that  $\mathcal{T}_K$  is a regular isotopy invariant. Thus we need check that  $\mathcal{T}_K$  is invariant under the moves  $RI$  and  $RI'$  on disoriented diagrams. Since  $cw(I) = \varepsilon(o) - \varepsilon(d) = 1 - 0 = 1$  and  $cw(I') = 0 - (-1) = 1$ , and so  $cw(I) = 1 + cw(I_0)$  and  $cw(I') = 1 + cw(I_1)$ , where  $I, I', I_0$  and  $I_1$  are diagrams given in Fig. 4, this follows at once. Indeed,

$$\begin{aligned} \mathcal{T}_I(A) &= (-A^3)^{-cw(I)} \langle I \rangle & \mathcal{T}_{I'}(A) &= (-A^3)^{-cw(I')} \langle I' \rangle \\ &= (-A^3)^{-(1+cw(I_0))} (-A^3) \langle I_0 \rangle & &= (-A^3)^{-(1+cw(I_1))} (-A^3) \langle I_1 \rangle \\ &= (-A^3)^{-cw(I_0)} \langle I_0 \rangle & &= (-A^3)^{-cw(I_1)} \langle I_1 \rangle \\ &= \mathcal{T}_{I_0}(A) & &= \mathcal{T}_{I_1}(A). \end{aligned} \quad \square$$

Finally, we have the particularly important behavior of the polynomials  $\langle K \rangle, f_K$  and  $\mathcal{T}_K$  under mirror images:

**Proposition 5.5.** *Let  $K^*$  denotes the mirror image of the (disoriented) knot and link  $K$  that is obtained by switching all the crossing of  $K$ . Then  $\langle K^* \rangle(A) = \langle K \rangle(A^{-1}), f_{K^*}(A) = f_K(A^{-1})$  and  $\mathcal{T}_{K^*}(A) = \mathcal{T}_K(A^{-1})$ .*

*Proof.* Reserving all crossings exchanges the roles of  $A$  and  $A^{-1}$  in the definition of  $\langle K \rangle, f_K$  and  $\mathcal{T}_K$ . □

Next, we show that  $\mathcal{T}_K$  is the Jones polynomial via the complete writhe. We call this polynomial as complete Jones polynomial for the disoriented knot and link diagrams and denote  $\mathcal{V}_K$ .

**Theorem 5.6.** *The complete normalize polynomial  $\mathcal{T}_K$  yields the complete Jones polynomial,  $\mathcal{V}_K(t)$ . That is,  $\mathcal{T}_K(t^{-1/4}) = \mathcal{V}_K(t)$ .*

*Proof.* Since the complete writhes of all the disoriented diagrams of  $K$  are equal by Theorem 4.3, we have that the complete normalized polynomial are equal for all choice of disorientation. From Remark 4.4, we obtain that the complete normalized polynomial of  $K$  is equal to the Kauffman polynomial of  $K$ . By Theorem 5.2. in [5],  $\mathcal{T}_K(t^{-1/4}) = \mathcal{V}_K(t)$ . □

**Example 5.7.** *Let  $\bigcirc_1$  and  $\bigcirc_2$  denotes the oriented and the disoriented diagrams of the unknot illustrated in Fig. 13, respectively. Then,*

$$\begin{aligned} \langle \bigcirc_1 \rangle &= -A^3, & \langle \bigcirc_2 \rangle &= -A^3, \\ f_{\bigcirc_1} &= (-A^3)^{-w(\bigcirc_1)} \langle \bigcirc_1 \rangle = 1, & f_{\bigcirc_2} &= (-A^3)^{-w(\bigcirc_2)} \langle \bigcirc_2 \rangle = A^6, \\ \mathcal{T}_{\bigcirc_1} &= (-A^3)^{-cw(\bigcirc_1)} \langle \bigcirc_1 \rangle = 1, & \mathcal{T}_{\bigcirc_2} &= (-A^3)^{-cw(\bigcirc_2)} \langle \bigcirc_2 \rangle = 1. \end{aligned}$$

With  $A = t^{-1/4}, V_{\bigcirc_2} = t^{-3/2}$  and  $\mathcal{V}_{\bigcirc_2} = 1$ .

**Fig. 13.** An Oriented and a Disoriented Unknot Diagram With One Crossing.



**Example 5.8.** *For the disoriented diagrams of the right hand trefoil in Fig. 3,*

$$\langle D_i \rangle = A^{-7} - A^{-3} - A^5, \quad i \in \{0, 1, 2, 3\},$$

$$\begin{aligned}
f_{D_0} &= (-A^3)^{-w(D_0)} \langle D_0 \rangle = A^{-4} + A^{-12} - A^{-16}, \\
f_{D_1} &= A^2 + A^{-6} - A^{-10} = A^6(A^{-4} + A^{-12} - A^{-16}), \\
f_{D_2} &= A^8 + 1 - A^{-4} = A^{12}(A^{-4} + A^{-12} - A^{-16}), \\
f_{D_3} &= A^{14} + A^6 - A^2 = A^{18}(A^{-4} + A^{-12} - A^{-16}).
\end{aligned}$$

With  $A = t^{-1/4}$ ,

$$\begin{aligned}
V_{D_0} &= t + t^3 - t^4, \\
V_{D_1} &= t^{-1/2} + t^{3/2} - t^{5/2} = t^{-3/2}(t + t^3 - t^4), \\
V_{D_2} &= t^{-2} + 1 - t = t^{-3}(t + t^3 - t^4), \\
V_{D_3} &= t^{-7/2} + t^{-3/2} - t^{-1/2} = t^{9/2}(t + t^3 - t^4).
\end{aligned}$$

$$\mathcal{T}_K = \mathcal{T}_{D_i} = (-A^3)^{-cw(D_i)} \langle D_i \rangle = A^{-4} + A^{-12} - A^{-16} \text{ and } \mathcal{V}_K = t + t^3 - t^4.$$

## 6 Discussion

In this paper, we have given an introduction to the subject of the disoriented knot theory and we have explained how to adapt the Reidemeister moves, the linking number, the writhe, the bracket polynomial and the Jones polynomial to the disoriented knot and link diagrams. We have based our work on disorientation. We have defined a new writhe called complete writhe and redefined the Jones polynomial for the disoriented knots and links diagrams. This paper lays the foundation for future works on these ideas.

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