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A Study on Dual Hyperbolic Fibonacci and Lucas Numbers

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Abstract

In this study, the dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers are introduced. Then, the fundamental identities are proven for these numbers. Additionally, we give the identities regarding negadual-hyperbolic Fibonacci and negadual-hyperbolic Lucas numbers. Finally, Binet formulas, D'Ocagne, Catalan and Cassini identities are obtained for dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers.

1 Introduction

Since the second half of 20th century, Golden section and Fibonacci numbers have received considerable attention by the researchers. Golden section firstly emerged in Euclid's Elements as an extreme division of line segment and mean ratio problem. The following algebraic equation was obtained in order to find the solution of this problem:

$$x^2 - x - 1 = 0.$$

Thus, the above equation has two roots

$$x_1 = \alpha = \frac{1+\sqrt{5}}{2}$$

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and

$$x_2 = -\frac{1}{\alpha} = \frac{1-\sqrt{5}}{2}$$

the positive root $x_1 = \alpha = \frac{1+\sqrt{5}}{2}$ is known as golden number. On the other hand, the Fibonacci numbers are determined by [12]

$$F_n = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

which is a numerical sequence, and is given by the following recurrence relation for $n \geq 1$ and the seeds $F_0 = 0,\ F_1 = 1$

$$F_{n+1} = F_n + F_{n-1}.$$

Similar to Fibonacci numbers, Lucas numbers are defined by Francois Edouard Anatole Lucas. Thus the Lucas numbers are determined by [12]

$$L_n = \{2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots\}$$

which is a numerical sequence, and is given by the following recurrence relation for $n \geq 1$ and the seeds $L_0 = 2,\ L_1 = 1$

$$L_{n+1} = L_n + L_{n-1}.$$

One of the important identities of Fibonacci numbers was Cassini identity which was obtained as follows by French mathematician Giovanni Domenico Cassini [4]

$$F_n^2 - F_{n-1}\, F_{n+1} = (-1)^{n+1}.$$

This identity connected the three arbitrary adjacent Fibonacci numbers as in $F_{n-1},\ F_n$ and F_{n+1}. The Cassini identity(for $r = 1$) is known as the special case of Catalan identity

$$F_n^2 - F_{n+r}\, F_{n-r} = (-1)^{n-r}\, F_r^2$$

which was discovered by Eugene Charles Catalan in 1879, [12]. On the other hand, French mathematician Jacques Philippe Marie Binet derived two remarkable formulas which connected the Fibonacci and Lucas numbers with the golden ratio. These formulas were given by

$$F_n = \frac{\alpha^n - (-1)^n\, \alpha^{-n}}{\sqrt{5}} \quad , \quad L_n = \alpha^n + (-1)^n\, \alpha^{-n}$$

and are called Binet formulas, [12].
The complex numbers have the form $x+iy$, where x and y are real numbers and

i is the imaginary unit. Taking into consideration this number system, several studies have been conducted with respect to complex Fibonacci numbers and complex Fibonacci quaternions [6, 8, 10]. Moreover, Nurkan and Güven have obtained some identities and formulas for bicomplex Fibonacci and Lucas numbers such as Cassini, Catalan identities and Binet formulas [15]. Analogously to the complex number, the hyperbolic number is $z = x + jy$, where x, y are two real numbers and j is called the hyperbolic imaginary unit such that $j^2 = 1$ and $j \notin R$. These numbers are also known as split-complex numbers, double numbers, perplex numbers, duplex numbers. At the end of the 20th century, Oleg Bodnar, Alexey Stakhov and Ivan Tkachenko revealed a new class of hyperbolic functions with the help of Golden ratio [1, 16]. Later, Stakhov and Rozin developed symmetrical hyperbolic Fibonacci and Lucas functions based on this theory [17]. After these studies Oleg Bodnar found the golden hyperbolic functions which led to using of these functions at the geometric theory of phyllotaxis (Bordnar's geometry). There was an analogy between the Binet formulas and hyperbolic functions. Thus, this new discovery resulted in a new class of hyperbolic functions which were named as hyperbolic Fibonacci and Lucas functions. Fibonacci and Lucas number theory has a direct analogy with the hyperbolic Fibonacci and Lucas functions. For the discrete values of the variable x, Fibonacci and Lucas numbers coincide with the hyperbolic Fibonacci and Lucas functions. Hence, we have described dual-complex Fibonacci, dual-complex Lucas numbers and have obtained the well-known identities for them [7].

We have introduced dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers. Then we have defined i-modulus of these numbers. While we are describing these moduli, the properties of the dual unit ε and the hyperbolic imaginary unit j have been considered. Thus, some identities with respect to dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers have been derived. The well-known identities have been used during these operations. Furthermore, Binet formulas have been obtained for these numbers. Finally, theorems consisting of negadual-hyperbolic Fibonacci and Lucas numbers and Catalan, Cassini, D'Ocagne identities for dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers have been stated.

2 Dual-Hyperbolic Fibonacci and Lucas Numbers

We will define the dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers. Then, some algebraic properties of dual-hyperbolic Fibonacci numbers will be mentioned. Finally, we will obtain some well-known identities and formulas involving dual-hyperbolic Fibonacci and Lucas numbers.

Definition 1. *The dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers are defined by*

$$DHF_n = F_n + F_{n+1}\,j + F_{n+2}\,\varepsilon + F_{n+3}\,j\varepsilon \tag{1}$$

and

$$DHL_n = L_n + L_{n+1}\,j + L_{n+2}\,\varepsilon + L_{n+3}\,j\varepsilon \tag{2}$$

respectively. Here F_n and L_n are the n^{th} Fibonacci and Lucas numbers. ε denotes the pure dual unit ($\varepsilon^2 = 0,\ \varepsilon \neq 0$), j denotes the hyperbolic unit $\big(j^2 = 1\big)$ and $j\varepsilon$ denotes the hyperbolic dual unit $\big((j\varepsilon)^2 = 0\big)$.

The set of the dual-hyperbolic Fibonacci numbers is represented as

$$\begin{aligned} DHF = \{\, & DHF_n = F_n + F_{n+1}\,j + F_{n+2}\,\varepsilon + F_{n+3}\,j\varepsilon | \\ & F_n\,is\,n^{th}\,\text{Fibonacci number},\ j^2 = 1,\ \varepsilon^2 = 0,\ (j\varepsilon)^2 = 0\Big\} \end{aligned}$$

The base elements $(1, j, \varepsilon, j\varepsilon)$ of dual-hyperbolic numbers correspond to the following commutative multiplications

$$j^2 = 1, \quad \varepsilon^2 = (j\varepsilon)^2 = 0, \quad \varepsilon\,(j\varepsilon) = (j\varepsilon)\,\varepsilon = 0, \quad j\,(j\varepsilon) = (j\varepsilon)\,j = \varepsilon.$$

Let DHF_n and DHF_m be two dual-hyperbolic Fibonacci numbers such as

$$DHF_n = F_n + F_{n+1}\,j + F_{n+2}\,\varepsilon + F_{n+3}\,j\varepsilon$$

and

$$DHF_m = F_m + F_{m+1}\,j + F_{m+2}\,\varepsilon + F_{m+3}\,j\varepsilon.$$

Then the addition and substraction of the dual-hyperbolic Fibonacci numbers are defined by

$$DHF_n \mp DHF_m = (F_n \mp F_m) + (F_{n+1} \mp F_{m+1})\,j + (F_{n+2} \mp F_{m+2})\,\varepsilon + (F_{n+3} \mp F_{m+3})\,j\varepsilon. \tag{3}$$

Multiplication of the two dual-hyperbolic Fibonacci numbers is given by

$$\begin{aligned} DHF_n \times DHF_m = {} & F_nF_m + F_{n+1}F_{m+1} + (F_{n+1}F_m + F_nF_{m+1})\,j \\ & + (F_nF_{m+2} + F_{n+1}F_{m+3} + F_{n+2}F_m + F_{n+3}F_{m+1})\,\varepsilon \\ & + (F_{n+1}F_{m+2} + F_nF_{m+3} + F_{n+3}F_m + F_{n+2}F_{m+1})\,j\varepsilon. \end{aligned} \tag{4}$$

When dual-hyperbolic Fibonacci number is considered as $DHF_n = (F_n + F_{n+1}j) + (F_{n+2} + F_{n+3}j)\,\varepsilon$, we come across five different conjugations as follow:

$$\begin{array}{lll} DHF_n^{\dagger_1} = (F_n - F_{n+1}j) + (F_{n+2} - F_{n+3}j)\,\varepsilon, & & \text{hyperbolic conjugation} \\ DHF_n^{\dagger_2} = (F_n + F_{n+1}j) - (F_{n+2} + F_{n+3}j)\,\varepsilon, & & \text{dual conjugation} \\ DHF_n^{\dagger_3} = (F_n - F_{n+1}j) - (F_{n+2} - F_{n+3}j)\,\varepsilon, & & \text{coupled conjugation} \\ DHF_n^{\dagger_4} = (F_n - F_{n+1}j) - \left(1 - \frac{F_{n+2}+F_{n+3}j}{F_n+F_{n+1}j}\varepsilon\right), & & \text{dual} - \text{hyperbolic conjugation} \\ DHF_n^{\dagger_5} = (F_{n+2} + F_{n+3}j) - (F_n - F_{n+1}j)\,\varepsilon, & & \text{anti} - \text{dual conjugation.} \end{array} \tag{5}$$

Now, we will obtain some equalities by using the algebraic properties of dual-hyperbolic Fibonacci numbers.

Proposition 1. *For any dual-hyperbolic Fibonacci number $DHF_n \in DHF$, we have*

1. $DHF_n + DHF_n^{\dagger_1} = 2\,(F_n + F_{n+2}\varepsilon) \in DF$
   $DHF_n \times DHF_n^{\dagger_1} = -F_{n+2}F_{n-1} \in DF$(Dual Fibonacci Number)
2. $DHF_n + DHF_n^{\dagger_2} = 2\,(F_n + F_{n+1}j) \in HF$
   $DHF_n \times DHF_n^{\dagger_2} = F_{2n+1} + 2F_nF_{n+1}j \in HF$(Hyperbolic Fibonacci Number)
3. $DHF_n + DHF_n^{\dagger_3} = 2\,(F_n + F_{n+3})\,j\varepsilon \in DHF$
   $DHF_n \times DHF_n^{\dagger_3} = -F_{n+2}F_{n-1} + 4\,(-1)^n\,j\varepsilon \in DHF$(Dual − Hyperbolic Fibonacci Number)
4. $DHF_n \times DHF_n^{\dagger_4} = F_n^2 - F_{n+1}^2 \in F$(Fibonacci Number)
5. $DHF_n \times DHF_n^{\dagger_5} = F_{2n+3} + (F_nF_{n+3} + F_{n+1}F_{n+2})j + (F_{2n+5} - F_nF_{n+2})\varepsilon$
   $+2F_{n+3}F_{n+2}j\varepsilon \in DHF$ (Dual − Hyperbolic Fibonacci Number)

6. $DHF_n - DHF_{n+1}j + DHF_{n+2}\varepsilon - DHF_{n+3}j\varepsilon = -F_{n+1}$.

Definition 2. *Let DHF_n be a dual-hyperbolic Fibonacci number. The i-modulus $(i = 1, 2, 3, 4, 5)$ of DHF_n are defined as follows*

$$DHF_n = F_n + F_{n+1}\,j + F_{n+2}\,\varepsilon + F_{n+3}\,j\varepsilon \tag{6}$$

and

$$\begin{aligned} |DHF_n|_1^2 &= DHF_n \times DHF_n^{\dagger_1} \\ |DHF_n|_2^2 &= DHF_n \times DHF_n^{\dagger_2} \\ |DHF_n|_3^2 &= DHF_n \times DHF_n^{\dagger_3} \\ |DHF_n|_4^2 &= DHF_n \times DHF_n^{\dagger_4} \\ |DHF_n|_5^2 &= DHF_n \times DHF_n^{\dagger_5}. \end{aligned} \tag{7}$$

Thus, the following theorem can be given.

Theorem 1. *Let DHF_n and DHL_n be a dual-hyperbolic Fibonacci number and a dual-hyperbolic Lucas number, respectively. In this case, for $n \geq 0$ we*

can give the following relations:
1. $DHF_n + DHF_{n+1} = DHF_{n+2}$
2. $DH\mathrm{L}_n + DH\mathrm{L}_{n+1} = DH\mathrm{L}_{n+2}$
3. $DHF_{n-1} + DHF_{n+1} = DH\mathrm{L}_\mathrm{n}$
4. $DHF_{n+2} - DHF_{n-2} = DH\mathrm{L}_\mathrm{n}$
5. $DHF_n^2 + DHF_{n+1}^2 = DHF_{2n+1} + F_{2n+3} + F_{2n+2}j + (2F_{2n+5} + F_{2n+3})\varepsilon + 3F_{2n+4}j\varepsilon$
6. $DHF_{n+1}^2 - DHF_{n-1}^2 = DHF_{2n} + F_{2n+2} + F_{2n+1}j + (F_{2n+2} + 2F_{2n+4})\varepsilon + 3F_{2n+3}j\varepsilon$
7. $DHF_n \times DHF_m + DHF_{n+1} \times DHF_{m+1} = DHF_{m+n+1} + F_{n+m+3} + F_{n+m+2}j + (F_{n+m+3} + 2F_{n+m+5})\,\varepsilon + 3F_{n+m+4}j\varepsilon.$

Proof of identity 1. By the Definition 1 and equation (3), we have

$$\begin{aligned} DHF_n + DHF_{n+1} &= (F_n + F_{n+1}j + F_{n+2}\varepsilon + F_{n+3}j\varepsilon) \\ &\quad + (F_{n+1} + F_{n+2}j + F_{n+3}\varepsilon + F_{n+4}j\varepsilon) \\ &= (F_n + F_{n+1}) + (F_{n+1} + F_{n+2})\,j \\ &\quad + (F_{n+2} + F_{n+3})\,\varepsilon + (F_{n+3} + F_{n+4})\,j\varepsilon \\ &= F_{n+2} + F_{n+3}j + F_{n+4}\varepsilon + F_{n+5}j\varepsilon \\ &= DHF_{n+2}. \end{aligned} .$$

Every dual-hyperbolic Fibonacci number is obtained by adding the last two dual-hyperbolic Fibonacci numbers to get the next one as in Fibonacci numbers.

Proof of identity 2. In the same manner to dual-hyperbolic Fibonacci numbers, we acquire

$$DH\mathrm{L}_n + DH\mathrm{L}_{n+1} = DH\mathrm{L}_{n+2}.$$

$$\begin{aligned} DH\mathrm{L}_n + DH\mathrm{L}_{n+1} &= (L_n + L_{n+1}j + L_{n+2}\varepsilon + L_{n+3}j\varepsilon) \\ &\quad + (L_{n+1} + L_{n+2}j + L_{n+3}\varepsilon + L_{n+4}j\varepsilon) \\ &= (L_n + L_{n+1}) + (L_{n+1} + L_{n+2})\,j \\ &\quad + (L_{n+2} + L_{n+3})\,\varepsilon + (L_{n+3} + L_{n+4})\,j\varepsilon \\ &= L_{n+2} + L_{n+3}j + L_{n+4}\varepsilon + L_{n+5}j\varepsilon \\ &= DH\mathrm{L}_{n+2}. \end{aligned}$$

Proofs of identities 3. and 4. Using the identities $F_{n+2} - F_{n-2} = L_n$, $F_{n+1} + F_{n-1} = L_n$ (see [18]) and equation (6) result in

$$\begin{aligned} DHF_{n-1} + DHF_{n+1} &= (F_{n-1} + F_nj + F_{n+1}\varepsilon + F_{n+2}j\varepsilon) \\ &\quad + (F_{n+1} + F_{n+2}j + F_{n+3}\varepsilon + F_{n+4}j\varepsilon) \\ &= (F_{n-1} + F_{n+1}) + (F_n + F_{n+2})\,j \\ &\quad + (F_{n+1} + F_{n+3})\,\varepsilon + (F_{n+2} + F_{n+4})\,j\varepsilon \\ &= L_n + L_{n+1}j + L_{n+2}\varepsilon + L_{n+3}j\varepsilon \\ &= DH\mathrm{L}_\mathrm{n} \end{aligned} .$$

and

$$\begin{aligned} DHF_{n+2} - DHF_{n-2} &= \left(F_{n+2} + F_{n+3}j + F_{n+4}\varepsilon + F_{n+5}j\varepsilon\right) \\ &\quad - \left(F_{n-2} + F_{n-1}j + F_{n}\varepsilon + F_{n+1}j\varepsilon\right) \\ &= \left(F_{n+2} - F_{n-2}\right) + \left(F_{n+3} - F_{n-1}\right)j \\ &\quad + \left(F_{n+4} - F_{n}\right)\varepsilon + \left(F_{n+5} - F_{n+1}\right)j\varepsilon \\ &= L_{n} + L_{n+1}j + L_{n+2}\varepsilon + L_{n+3}j\varepsilon \\ &= DH\mathrm{L_n}. \end{aligned}$$

Thus, the proofs of identities 3. and 4. are completed.

Proof of identity 5. Equation (4) gives us

$$\begin{aligned} DHF_n^2 = {} & F_n^2 + F_{n+1}^2 + 2F_nF_{n+1}j + 2\left(F_nF_{n+2} + F_{n+1}F_{n+3}\right)\varepsilon \\ & +2\left(F_nF_{n+3} + F_{n+1}F_{n+2}\right)j\varepsilon \end{aligned}$$

and

$$\begin{aligned} DHF_{n+1}^2 = {} & F_{n+1}^2 + F_{n+2}^2 + 2F_{n+1}F_{n+2}j + 2\left(F_{n+1}F_{n+3} + F_{n+2}F_{n+4}\right)\varepsilon \\ & +2\left(F_{n+1}F_{n+4} + F_{n+2}F_{n+3}\right)j\varepsilon. \end{aligned}$$

As a result, using the identities $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$ and $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ (see [18]), the following identity can be found

$$DHF_n^2 + DHF_{n+1}^2 = DHF_{2n+1} + F_{2n+3} + F_{2n+2}j + (2F_{2n+5} + F_{2n+3})\varepsilon + 3F_{2n+4}j\varepsilon.$$

Thus, the identity 5. is proved.

Proofs of identities 6. and 7. Considering the equations (3), (4) and applying the identities $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$ and $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ (see [18]), we can conclude

$$\begin{aligned} DHF_{n+1}^2 - DHF_{n-1}^2 &= \big[F_{n+1}^2 + F_{n+2}^2 + 2F_{n+1}F_{n+2}j + 2\left(F_{n+1}F_{n+3} + F_{n+2}F_{n+4}\right)\varepsilon \\ &\qquad + 2\left(F_{n+1}F_{n+4} + F_{n+2}F_{n+3}\right)j\varepsilon\big] \\ &\quad - \big[F_{n-1}^2 + F_n^2 + 2F_{n-1}F_nj + 2\left(F_{n-1}F_{n+1} + F_nF_{n+2}\right)\varepsilon \\ &\qquad + 2\left(F_{n-1}F_{n+2} + F_nF_{n+1}\right)j\varepsilon\big] \\ &= DHF_{2n} + F_{2n+2} + F_{2n+1}j + (F_{2n+2} + 2F_{2n+4})\varepsilon + 3F_{2n+3}j\varepsilon \end{aligned}$$

and

$$\begin{aligned} & DHF_n \times DHF_m + DHF_{n+1} \times DHF_{m+1} \\ &\quad = F_nF_m + F_{n+1}F_{m+1} + \left(F_{n+1}F_m + F_nF_{m+1}\right)j \\ &\qquad + \left(F_nF_{m+2} + F_{n+1}F_{m+3} + F_{n+2}F_m + F_{n+3}F_{m+1}\right)\varepsilon \\ &\qquad + \left(\mathrm{F_{n+1}F_{m+2} + F_nF_{m+3} + F_{n+3}F_m + F_{n+2}F_{m+1}}\right)\mathrm{j}\varepsilon \\ &\qquad + F_{n+1}F_{m+1} + F_{n+2}F_{m+2} + \left(F_{n+2}F_{m+1} + F_{n+1}F_{m+2}\right)j \\ &\qquad + \left(F_{n+1}F_{m+3} + F_{n+2}F_{m+4} + F_{n+3}F_{m+1} + F_{n+4}F_{m+2}\right)\varepsilon \\ &\qquad + \left(F_{n+2}F_{m+3} + F_{n+1}F_{m+4} + F_{n+4}F_{m+1} + F_{n+3}F_{m+2}\right)j\varepsilon \\ &\quad = DHF_{m+n+1} + F_{n+m+3} + F_{n+m+2}j + \left(F_{n+m+3} + 2F_{n+m+5}\right)\varepsilon \\ &\qquad + 3F_{n+m+4}j\varepsilon. \end{aligned}$$

Now, we will give D'Ocagne's identity which is known as one of the determinantal identities for Fibonacci numbers.

Theorem 2. *For $n, m \geq 0$, the D'Ocagne identity of the dual-hyperbolic Fibonacci numbers DHF_n and DHF_m is given by*

$$DHF_m \times DHF_{n+1} - DHF_{m+1} \times DHF_n = (-1)^n F_{m-n}(1 + j + 3j\varepsilon).$$

Proof. In order to prove the claim, we consider the equation (4). Thus, the following equations can be written

$$\begin{aligned} DHF_m \times DHF_{n+1} = {} & F_m F_{n+1} + F_{m+1} F_{n+2} + (F_{m+1} F_{n+1} + F_m F_{n+2})\, j \\ & + (F_m F_{n+3} + F_{m+1} F_{n+4} + F_{m+2} F_{n+1} + F_{m+3} F_{n+2})\, \varepsilon \\ & + (F_{m+1} F_{n+3} + F_m F_{n+4} + F_{m+3} F_{n+1} + F_{m+2} F_{n+2})\, j\varepsilon. \end{aligned} \tag{8}$$

and

$$\begin{aligned} DHF_{m+1} \times DHF_n = {} & F_{m+1} F_n + F_{m+2} F_{n+1} + (F_{m+2} F_n + F_{m+1} F_{n+1})\, j \\ & + (F_{m+1} F_{n+2} + F_{m+2} F_{n+3} + F_{m+3} F_n + F_{m+4} F_{n+1})\, \varepsilon \\ & + (F_{m+2} F_{n+2} + F_{m+1} F_{n+3} + F_{m+4} F_n + F_{m+3} F_{n+1})\, j\varepsilon. \end{aligned} \tag{9}$$

Substracting the equation (8) from equation (9), it follows that

$$DHF_m \times DHF_{n+1} - DHF_{m+1} \times DHF_n = (-1)^n F_{m-n}(1 + j + 3j\varepsilon).$$

Therefore, we find the desired result.

Theorem regarding negadual-hyperbolic Fibonacci and negadual-hyperbolic Lucas numbers is:

Theorem 3. *Let DHF_{-n} and DHL_{-n} be negadual-hyperbolic Fibonacci and negadual- hyperbolic Lucas numbers. For $n \geq 0$, the following identities are hold.*

1. $DHF_{-n} = (-1)^{n+1} DHF_n + (-1)^n L_n (j + \varepsilon + 2j\varepsilon)$

2. $DHL_{-n} = (-1)^n DHL_n + (-1)^{n-1} 5F_n (j + \varepsilon + 2j\varepsilon)$

Proof. If we use the Definition 1 for F_{-n} and the identities $F_n + F_{n+2} = L_{n+1}, \quad (-1)^{n+1} F_n = F_{-n}$(see [12, 11, 5]), then a direct calculation will show that

$$\begin{aligned}
DHF_{-n} &= F_{-n} + F_{-n+1}\, j + F_{-n+2}\, \varepsilon + F_{-n+3}\, j\varepsilon \\
&= (-1)^{n+1} F_n + (-1)^n F_{n-1}\, j + (-1)^{n+1} F_{n-2}\, \varepsilon + (-1)^n F_{n-3}\, j\varepsilon \\
&= (-1)^{n+1} F_n + (-1)^{n+1} F_{n+1}\, j + (-1)^{n+1} F_{n+2}\, \varepsilon + (-1)^{n+1} F_{n+3}\, j\varepsilon \\
&\quad - (-1)^{n+1} F_{n+1}\, j - (-1)^{n+1} F_{n+2}\, \varepsilon - (-1)^{n+1} F_{n+3}\, j\varepsilon \\
&\quad + (-1)^n F_{n-1}\, j + (-1)^{n+1} F_{n-2}\, \varepsilon + (-1)^n F_{n-3}\, j\varepsilon \\
&= (-1)^{n+1} DHF_n + (-1)^n \left[F_{n-1} + F_{n+1}\right] j + (-1)^n \left[F_{n+2} - F_{n-2}\right] \varepsilon \\
&\quad + (-1)^n \left[F_{n-3} + F_{n+3}\right] j\varepsilon \\
&= (-1)^{n+1} DHF_n + (-1)^n L_n\, j + (-1)^n L_n\, \varepsilon + (-1)^n 2L_n\, j\varepsilon \\
&= (-1)^{n+1} DHF_n + (-1)^n L_n \left(j + \varepsilon + 2j\varepsilon\right).
\end{aligned}$$

Again considering Definition 1 for L_{-n} and applying the identities $L_{-n} = (-1)^n L_n$, $L_{m+n} + L_{m-n} = \begin{cases} 5F_m F_n, & n = 2k+1 \\ L_m L_n, & n \neq 2k+1 \end{cases}$ (see [11], [12]), we get

$$\begin{aligned}
DHL_{-n} &= L_{-n} + L_{-n+1}\, j + L_{-n+2}\, \varepsilon + L_{-n+3} j\varepsilon \\
&= (-1)^n L_n + (-1)^{n-1} L_{n-1}\, j + (-1)^{n-2} L_{n-2}\, \varepsilon + (-1)^{n-3} L_{n-3}\, j\varepsilon \\
&= (-1)^n L_n + (-1)^n L_{n+1}\, j + (-1)^n L_{n+2}\, \varepsilon + (-1)^n L_{n+3}\, j\varepsilon \\
&\quad - (-1)^n L_{n+1}\, j - (-1)^n L_{n+2}\, \varepsilon - (-1)^n L_{n+3}\, j\varepsilon \\
&\quad + (-1)^{n-1} L_{n-1}\, j + (-1)^{n-2} L_{n-2}\, \varepsilon + (-1)^{n-3} L_{n-3}\, j\varepsilon \\
&= (-1)^{n+1} DHL_n + (-1)^{n-1} \left[L_{n-1} + L_{n+1}\right] j + (-1)^{n-2} \left[L_{n+2} - L_{n-2}\right] \varepsilon \\
&\quad + (-1)^{n-1} \left[L_{n-3} + L_{n+3}\right] j\varepsilon \\
&= (-1)^{n+1} DHL_n + 5\,(-1)^{n-1} F_n\, j + 5\,(-1)^{n-1} F_n\, \varepsilon + 10\,(-1)^n F_n\, j\varepsilon \\
&= (-1)^n DHL_n + (-1)^{n-1} 5F_n \left(j + \varepsilon + 2j\varepsilon\right).
\end{aligned}$$

Theorem 4 (Binet's Identity)**.** *Let DHF_n and DHL_n be a dual-hyperbolic Fibonacci number and a dual-hyperbolic Lucas number, respectively. For $n \geq 1$, the Binet's formulas for these dual-hyperbolic numbers are expressed as follow:*

$$DHF_n = \frac{\bar{\alpha}\, \alpha^n - \bar{\beta}\, \beta^n}{\alpha - \beta}$$

and

$$DHL_n = \bar{\alpha}\, \alpha^n + \bar{\beta}\, \beta^n$$

where $\bar{\alpha} = 1 + \alpha j + \alpha^2 \varepsilon + \alpha^3 j\varepsilon$ and $\bar{\beta} = 1 + \beta j + \beta^2 \varepsilon + \beta^3 j\varepsilon$.

Proof. By using the Binet's formulas for the Fibonacci and Lucas numbers, by a direct calculation one can find that

$$\begin{aligned}
DHF_n &= F_n + F_{n+1} j + F_{n+2} \varepsilon + F_{n+3} j\varepsilon \\
&= \tfrac{\alpha^n - \beta^n}{\alpha - \beta} + \tfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} j + \tfrac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} \varepsilon + \tfrac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} j\varepsilon \\
&= \tfrac{\alpha^n \left(1 + \alpha j + \alpha^2 \varepsilon + \alpha^3 j\varepsilon\right) - \beta^n \left(1 + \beta j + \beta^2 \varepsilon + \beta^3 j\varepsilon\right)}{\alpha - \beta}
\end{aligned}$$

and

$$\begin{aligned} DHL_n &= L_n + L_{n+1}j + L_{n+2}\varepsilon + L_{n+3}j\varepsilon \\ &= \alpha^n + \beta^n + \left(\alpha^{n+1} + \beta^{n+1}\right) j + \left(\alpha^{n+2} + \beta^{n+2}\right) \varepsilon + \left(\alpha^{n+3} + \beta^{n+3}\right) j\varepsilon \\ &= \alpha^n \left(1 + \alpha i + \alpha^2\varepsilon + \alpha^3 j\varepsilon\right) + \beta^n \left(1 + \beta j + \beta^2\varepsilon + \beta^3 j\varepsilon\right) \end{aligned}$$

Finally, putting $\bar{\alpha}$ for $1 + \alpha j + \alpha^2\varepsilon + \alpha^3 j\varepsilon$ and $\bar{\beta}$ for $1 + \beta j + \beta^2\varepsilon + \beta^3 j\varepsilon$, it is easily seen that

$$DHF_n = \frac{\bar{\alpha}\,\alpha^n - \bar{\beta}\,\beta^n}{\alpha - \beta}$$

and

$$DHL_n = \bar{\alpha}\,\alpha^n + \bar{\beta}\,\beta^n$$

for dual-hyperbolic Fibonacci and Lucas numbers, respectively.

Theorem 5 (Cassini's Identities)**.** *Let DHF_n and DHL_n be a dual-hyperbolic Fibonacci number and a dual-hyperbolic Lucas number, respectively. For $n \geq 1$, the following identities are the Cassini's Identities for DHF_n and DHL_n* ***1.*** $DHF_{n+1} \times DHF_{n-1} - DHF_n^2 = (-1)^n (j + 3j\varepsilon)$
2. $DHL_{n+1} \times DHL_{n-1} - DHL_n^2 = 5\,(-1)^{n-1} (j + 3j\varepsilon)$.

Proof of identity 1. Applying the equations (3), (4) and arranging the terms, the expression $DHF_{n+1} \times DHF_{n-1} - DHF_n^2$ becomes

$$\begin{aligned} DHF_{n+1} \times DHF_{n-1} - DHF_n^2 = {} & [F_{n+1}F_{n-1} + F_{n+2}F_n + (F_{n+2}F_{n-1} + F_{n+1}F_n)\, j \\ & + \left(F_{n+1}^2 + F_{n+2}^2 + F_{n+3}F_{n-1} + F_{n+4}F_n\right) \varepsilon \\ & + (2F_{n+1}F_{n+2} + F_{n-1}F_{n+4} + F_nF_{n+3})\, j\varepsilon] \\ & - \left[F_n^2 + F_{n+1}^2 + 2F_nF_{n+1}j + 2(F_{n+2}F_n + F_{n+3}F_{n+1})\varepsilon \right. \\ & \left. + 2(F_nF_{n+3} + F_{n+2}F_{n+1})j\varepsilon\right]. \end{aligned}$$

Using the identities of Fibonacci numbers $F_m\,F_{n+1} - F_{m+1}\,F_n = (-1)^n\,F_{m-n}$, $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$, $F_n\,F_m + F_{n+1}\,F_{m+1} = F_{m+n+1}$ and $F_{-n} = (-1)^{n+1}\,F_n$ (see [12, 19, 11, 18]) lead to

$$DHF_{n+1} \times DHF_{n-1} - DHF_n^2 = (-1)^n (j + 3j\varepsilon)\,.$$

Proof of identity 2. According to addition and multiplication of two dual-hyperbolic Lucas numbers, we see that

$$\begin{aligned} DHL_{n+1} \times DHL_{n-1} - DHL_n^2 = {} & [L_{n+1}L_{n-1} + L_{n+2}L_n + (L_{n+2}L_{n-1} + L_{n+1}L_n)\, j \\ & + \left(L_{n+1}^2 + L_{n+2}^2 + L_{n+3}L_{n-1} + L_{n+4}L_n\right) \varepsilon \\ & + (2L_{n+1}L_{n+2} + L_{n-1}L_{n+4} + L_nL_{n+3})\, j\varepsilon] \\ & - \left[L_n^2 + L_{n+1}^2 + 2L_nL_{n+1}j + 2(L_{n+2}L_n + L_{n+3}L_{n+1})\varepsilon \right. \\ & \left. + 2(L_nL_{n+3} + L_{n+2}L_{n+1})j\varepsilon\right]. \end{aligned}$$

Repeating the similar calculations in previous proof of identity 1. and using the identity $L_{n-1}L_{n+1} - L_n^2 = 5\left(-1\right)^{n-1}$ (see [12]) in the above equation, the desired result is found as

$$DHL_{n+1} \times DHL_{n-1} - DHL_n^2 = 5\left(-1\right)^{n-1}(j + 3j\varepsilon).$$

Thus, the proof is completed.

Theorem 6 (Catalan's Identity). *The Catalan identity for the dual-hyperbolic Fibonacci numbers is given by*

$$DHF_n^2 - DHF_{n+r} \times \ DHF_{n-r} = \left(-1\right)^{n-r} F_r^2 \left(j + 3j\varepsilon\right).$$

Proof. Considering the the equations (3) and (4), we get

$$\begin{aligned}
&DHF_n^2 - F_{n-r} \times F_{n-r} \\
&\quad = \left[F_n^2 + F_{n+1}^2 + 2\left(F_{n+1}F_n\right)j + 2\left(F_{n+2}F_n + F_{n+1}F_{n+3}\right)\varepsilon \right.\\
&\qquad \left. + 2\left(F_nF_{n+3} + F_{n+1}F_{n+2}\right)j\varepsilon\right] \\
&\quad - [F_{n+r}F_{n-r} + F_{n+r+1}F_{n-r+1} + (F_{n+r}F_{n-r+1} + F_{n+r+1}F_{n+r})j \\
&\qquad + (F_{n+r}F_{n-r+2} + F_{n+r+2}F_{n-r} + F_{n+r+1}F_{n-r+3} + F_{n+r+3}F_{n-r+1})\varepsilon \\
&\qquad +(F_{n+r}F_{n-r+3} + F_{n+r+3}F_{n-r} + F_{n+r+1}F_{n-r+2} + F_{n+r+2}F_{n-r+1})j\varepsilon].
\end{aligned}$$

Putting the identities $F_n^2 - F_{n-r}\, F_{n+r} = \left(-1\right)^{n-r} F_r^2$ and $F_m\, F_n - F_{m+k}\, F_{n-k} = \left(-1\right)^{n-k} F_{m+k-n}\, F_k$ (see [19]) into the last equation, we obtain

$$DHF_n^2 - DHF_{n+r} \times \ DHF_{n-r} = \left(-1\right)^{n-r} F_r^2 \left(j + 3j\varepsilon\right).$$

3 Conclusions

When the literature is reviewed, it can be seen that several studies have been conducted on quaternions, split quaternions, complex quaternions, dual quaternions, hyperbolic quaternions, and one can find the results regarding these quaternions and their properties in [2], [3], [9], [13]. Here, the studies about these quaternions can be summarized as follows:
A generalized quaternion can be written in the following form

$$q = a_0 + a_1\, i + a_2\, j + a_3\, k$$

where the coefficients $a_0,\, a_1,\, a_2, a_3$ are real numbers and i, j, k represent the quaternionic units which satisfy the equalities

$$\begin{aligned}
&i^2 = -\alpha\,, \qquad && j^2 = -\beta\,, \quad && k^2 = -\alpha\,\beta \\
&ij = -ji = k\,, && jk = -kj = \beta i \quad \text{and} && ki = -ik = \alpha j
\end{aligned}$$

where $\alpha,\, \beta \in R$. Special cases can be seen at the following scheme according to choice of α and β

$\alpha = 1, \quad \beta = 1$	Real quaternion
$\alpha = 1, \quad \beta = -1$	Split quaternion
$\alpha = 1, \quad \beta = 0$	Semi-quaternion
$\alpha = -1, \quad \beta = 0$	Split semi-quaternion
$\alpha = 0, \quad \beta = 0$	$\frac{1}{4}$-quaternion

Horadam initially described Fibonacci quaternions taking the coefficients of a quaternion as Fibonacci numbers [10]. Recently, many authors have studied Fibonacci and Lucas quaternions based on this paper. Moreover, these studies have been extended to octonions.

Our paper is motivated by this question: What happens if the components of dual numbers become hyperbolic numbers? This idea led to the concept of dual-hyperbolic numbers with Fibonacci and Lucas coefficients. This number system is commutative and five different conjugations can be defined (see page 3). Therefore, we have achieved a result which includes Fibonacci numbers, hyperbolic Fibonacci numbers, dual Fibonacci numbers and dual-hyperbolic Fibonacci numbers, which can be seen in Proposition 1. Furthermore, this idea can be extended to eight-component number system joining the complex, hyperbolic and dual numbers such as

$$z = a + ib + jc + \mu d + ep + fq + gu + hv$$

where $1, i, j, \mu, p, q, u$ and v are the basis of the eight-component number. The multiplication scheme becomes [14]

$\times$	1	i	j	μ	p	q	u	v
1	1	i	j	μ	p	q	u	v
i	i	-1	p	q	$-j$	$-\mu$	v	$-u$
j	j	p	1	u	i	v	μ	q
μ	μ	q	u	0	v	0	0	0
p	p	$-j$	i	v	-1	$-u$	0	0
q	$-q$	$-\mu$	v	0	$-u$	0	0	0
u	u	v	μ	0	q	0	0	0
v	v	$-u$	q	0	$-\mu$	0	0	0

While the field of octonions is non-commutative and non-associative real field, this new number system becomes both commutative and associative.

The present study is useful for the study of mathematical models which are the classes of Fibonacci numbers, golden proportions, Binet formulas, Lucas numbers and golden matrices. Thus, we believe that these results will contribute to the algorithmic measurement theory, new computer arithmetic, new coding theory and the mathematical harmony.

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