CHARACTERIZATION OF SLANT HELIX İN GALILEAN AND PSEUDO-GALILEAN SPACES

Murat Kemal KARACAN * and Yılmaz TUNÇER **

*Usak University, Faculty of Sciences and Arts,Department of Mathematics,1 Eylul Campus,64200,Usak-TURKEY, murat.karacan@usak.edu.tr
**Usak University, Faculty of Sciences and Arts,Department of Mathematics,1 Eylul Campus,64200,Usak-TURKEY,yilmaz.tuncer@usak.edu.tr

ABSTRACT

We consider a curve $\alpha = \alpha(s)$ parameterized by the arc length s in Galilean and Pseudo-Galilean spaces and denote by $\{T, N, B\}$ the Frenet frame of $\alpha = \alpha(s)$. We say that is a slant helix if there exists a fixed direction U of G_3 and G_3^1 such that the functions $\langle N,U \rangle_{G_3}$ and $\langle N,U \rangle_{G_3^1}$ are constant. In this work we give characterizations of slant helices in terms of the curvature and torsion of α .

GALİLEAN VE PSEUDO-GALİLEAN UZAYLARINDA SLANT HELİSİN KARAKTERİZASYONU

ÖZET

Bu çalışmada, 3- boyutlu Galilean ve Pseudo Galilean uzaylarında yay parametreli ve $\{T, N, B\}$ Frenet çatısıyla verilen bir eğrinin, asli normali ile sabit bir doğrultu arasındaki açının sabit olmasını sağlayan slant helis olma durumunu, eğrinin eğrilik ve torsiyonu yardımıyla karakterize ettik.

1.INTRODUCTION

This definition is motivated by what happens in Euclidean space E^3. In this setting, we recall that a helix is a curve where the tangent lines make a constant angle with a fixed direction. Helices are characterized by the fact that the ratio $\frac{\tau}{\kappa}$ is constant along the curve [4,7]. Izumiya and Takeuchi have introduced the concept of Slant helix in Euclidean space by saying that the principal normal lines make a constant angle with a fixed direction [6].They characterize a slant helix if and only if the function

$$\frac{\kappa^2}{\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)' \tag{1.1}$$

is constant. See also [2,6,8].Recently, helices in Galilean space G_3 have been studied depending on the causal character of the curve α : see for example [1,3].

Thus, our definition of slant helix are the Galilean and Pseudo-Galilean versions of the Euclidean one. Our main results in this work is the following characterization of Slant helices in the spirit of the one given in equation (1.1). We will assume throughout this work that the curvature and torsion functions do not equal zero.

2.GALILEAN SPACE G_3

The Galilean space is a three dimensional complex projective space, P_3, in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points, $I_1, I_2 \in f$ (the absolute points).

We shall take, as a real model of the space G_3, a real projective space P_3, with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$ and a real line $f \subset w$, on which an elliptic involution ε has been defined. Let ε be in homogeneous coordinates

$$w...x_0 = 0, \qquad f...x_0 = 0$$
$$\varepsilon : (0:0:x_2:x_3) \to (0:0:x_3:-x_2).$$

In the nonhomogeneous coordinates, the similarity group H_8 has the form

$$\begin{aligned} \bar{x} &= a_{11} + a_{12}x \\ \bar{y} &= a_{21} + a_{22}x + a_{23}\cos\theta + a_{23}\sin\theta \\ \bar{z} &= a_{31} + a_{32}x - a_{23}\sin\theta + a_{23}\cos\theta \end{aligned} \tag{2.1}$$

where a_{ij} and θ are real numbers.For $a_{11} = a_{23} = 1$,we have have the subgroup B_6 , the group of Galilean motions:

$$\begin{aligned} \bar{x} &= a_{11} + a_{12}x \\ \bar{y} &= b + cx + y\cos\theta + z\sin\theta \\ \bar{z} &= d + ex - y\sin\theta + z\cos\theta \end{aligned}$$

In G_3, there are four classes of lines:

a) (proper) nonisotropic lines-they do not meet the absolute line f .

b) (proper) isotropic lines-lines that do not belong to the plane w but meet the absolute line f .

c) unproper nonisotropic lines-all lines of w but f .

d) the absolute line f .

Planes $x = constant$ are Euclidean and so is the plane w. Other planes are isotropic. In what follows, the coefficients a_{11} and a_{23} a will play a special role. In particular, for $a_{11} = a_{23} = 1$, (2.1) defines the group $B_6 \subset H_8$ of isometries of the Galilean space G_3.

The scalar product in Galilean space G_3 is defined by

$$\langle X,Y \rangle_{G_3} = \begin{cases} x_1y_1 & , \quad if \quad x_1 \neq 0 \quad or \quad y_1 \neq 0 \\ x_2y_2 + x_3y_3 & , \quad if \quad x_1 = 0 \quad and \quad y_1 = 0 \end{cases}$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$.

A curve $\alpha : I \subseteq R \to G_3$ of the class C^r $(r \geq 3)$ in the Galilean space G_3 is given defined by

$$\alpha(x) = (s, y(s), z(s)) \tag{2.2}$$

where s is a Galilean invariant and the arc length on α.The curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(s) = \sqrt{(y''(s))^2 + (z''(s))^2}\ ,\ \ \tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)} \tag{2.3}$$

The orthonormal frame in the sense of Galilean space G_3 is defined by

$$\begin{aligned} T &= \alpha'(s) = (1, y'(s), z'(s)) \\ N &= \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s)) \\ B &= \frac{1}{\kappa(s)}(0, -z''(s), y''(s)). \end{aligned} \tag{2.4}$$

The vectors T, N and B in (2.4) are called the vectors of the tangent, principal normal and the binormal line of α, respectively.They satisfy the following Frenet equations [1]

$$\begin{aligned} T' &= \kappa N \\ N' &= \tau B \\ B' &= -\tau N. \end{aligned} \tag{2.5}$$

3.PSEUDO-GALILEAN SPACE G_3^1

The geometry of the pseudo-Galilean space is similar (but not the same) to the Galilean space.The pseudo-Galilean space G_3^1 is a three-dimensional projective space in which the absolute consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and a hyperbolic involution on f. Projective transformations which presere the absolute form of a group H_8 and are in nonhomogeneous coordinates can be written in the form

$$\begin{aligned}\bar{x} &= a + bx \\ \bar{y} &= c + dx + r\,y\cosh + r\,z\sinh\theta \\ \bar{z} &= e + fx + r\,y\sinh\theta + r\,z\cosh\theta\end{aligned} \tag{3.1}$$

where a, b, c, d, e, f, r and θ are real numbers. Particularly, for $b = r = 1$, the group (3.1) becomes the group $B_6 \subset H_8$ of isometries (proper motions) of the pseudo-Galilean space G_3^1. The motion group leaves invariant the absolute figure and defines the other invariants of this geometry.It has the following form

$$\begin{aligned}\bar{x} &= a + x \\ \bar{y} &= c + dx + y\cosh + z\sinh\theta \\ \bar{z} &= e + fx + y\sinh\theta + z\cosh\theta.\end{aligned} \tag{3.2}$$

According to the motion group in the pseudo-Galilean space, there are nonisotropic vectors $X(x, y, z)$ (for which holds $x \neq 0$) and four types of isotropic vectors: spacelike $(x = 0, y^2 - z^2 > 0)$, timelike $(x = 0, y^2 - z^2 < 0)$ and two types of lightlike vectors $(x = 0, y = \pm z)$ .The scalar product of two vectors $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ in G_3^1 is defined by

$$\langle A, B\rangle_{G_3^1} = \begin{cases} a_1 b_1 & , \quad if \;\; a_1 \neq 0 \;\;\; or \;\; b_1 \neq 0 \\ a_2 b_2 - a_3 b_3 & , \quad if \;\; a_1 = 0 \;\;\; and \;\; b_1 = 0. \end{cases} \tag{3.3}$$

A curve $\alpha(t) = (x(t), y(t), z(t))$ is admissible if it has no inflection points, no isotropic tangents or tangents or normals whose projections on the absolute plane would be light-like vectors.For an admissible curve $\alpha : I \subseteq R \to G_3^1$ the curvature $\kappa(t)$ and the torsion $\tau(t)$ are defined by

$$\kappa(t) = \frac{\sqrt{(y''(t))^2 - (z''(t))^2}}{(x'(t))^2}, \quad \tau(t) = \frac{y''(t)z'''(t) - y'''(t)z''(t)}{|x'(t)|^5 \kappa^2(t)}. \tag{3.4}$$

expressed in components.Hence, for an admissible curve $\alpha : I \subseteq R \to G_3^1$ parameterized by the arc length s with differential form $ds = dx$, given by

$$\alpha(t) = \left(x, y(s), z(s)\right), \tag{3.5}$$

the formulas (3.4) have the following form

$$\kappa(s) = \sqrt{\left|\left(y''(s)\right)^2 - \left(z''(s)\right)^2\right|}\ , \quad \tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}. \tag{3.6}$$

The associated trihedron is given by

$$\begin{aligned} T &= \alpha'(s) = \left(1, y'(s), z'(s)\right) \\ N &= \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}\left(0, y''(s), z''(s)\right) \\ B &= \frac{1}{\kappa(s)}\left(0, \varepsilon z''(s), \varepsilon y''(s)\right). \end{aligned} \tag{3.7}$$

where $\varepsilon = \mp 1$, chosen by criterion $\det(T, N, B) = 1$, that means

$$\left|\left(y''(s)\right)^2 - \left(z''(s)\right)^2\right| = \varepsilon\left(\left(y''(s)\right)^2 - \left(z''(s)\right)^2\right).$$

The curve α given by (3.6) is timelike (resp. spacelike) if $N(s)$ is a spacelike(resp. timelike) vector. The principal normal vector or simply normal is spacelike if $\varepsilon = 1$ and timelike if $\varepsilon = -1$.For derivatives of the tangent (vector) T , the normal N and the binormal B ,respectively, the following Serret-Frenet formulas hold

$$\begin{aligned} T' &= \kappa N \\ N' &= \tau B \\ B' &= \tau N. \end{aligned} \tag{3.8}$$

From (3.8), we derive an important relation [8],

$$\alpha'''(s) = \kappa'(s)N(s) + \kappa(s)\tau(s)B(s). \tag{3.9}$$

4.SLANT HELICES IN G_3

Definition 4.1. A curve α is called a slant helix if there exists a constant vector field U in G_3 such that the function $\langle N(s),U\rangle_{G_3}$ is constant.

Theorem 4.1. Let α be a curve parameterized by the arc length s in G_3 .Then α is a slant helix if and only if either one the next two functions

$$\pm\frac{\kappa^2}{\tau^3}\left(\frac{\tau}{\kappa}\right)' \tag{4.1}$$

is constant everywhere τ does not vanish.

Proof. Let α be a curve in .In order to prove Theorem 4.1, we first assume that α is a slant helix. Let U be the vector field such that the function $\langle N(s),U\rangle_{G_3}=c$ is constant. There exist smooth functions a_1 and a_3 such that

$$U=a_1(s)T(s)+cN(s)+a_3(s)B(s) \tag{4.2}$$

As U is constant, a differentiation in (4.2) together (4.1) gives

$$\begin{aligned} a_1' &= 0 \\ a_1\kappa-a_3\tau &= 0 \\ a_3'+c\tau &= 0. \end{aligned} \tag{4.3}$$

From the second equation in (4.3) we have

$$a_1=a_3\left(\frac{\tau}{\kappa}\right). \tag{4.4}$$

Moreover, if $a_1\neq 0$,

$$\langle U,U\rangle_{G_3}=a_1^2=constant \tag{4.5}$$

We point out that this constraint, together the second and third equation of (4.3) is equivalent to the very system (4.3). From (4.4) and (4.5), set

$$a_3^2\left(\frac{\tau}{\kappa}\right)^2=m^2. \tag{4.6}$$

Thus , (4.6) which give

$$a_3 = \pm \frac{m}{\left(\frac{\tau}{\kappa}\right)}$$

on I . The third equation in (4.3) yields

$$\frac{d}{ds}\left(\pm \frac{m}{\left(\frac{\tau}{\kappa}\right)} \right) = -c\tau$$

on I . This can be written as

$$\frac{\kappa^2}{\tau^3}\left(\frac{\tau}{\kappa}\right)' = \pm \frac{c}{m}. \tag{4.7}$$

This shows a part of Theorem 4.1. Conversely, assume that the condition (4.1) is satisfied. In order to simplify the computations, we assume that the function in (4.1) is a constant, namely, c .We define

$$U = T + cN + \frac{\tau}{\kappa}B. \tag{4.8}$$

A differentiation of (4.8) together the Frenet equations in G_3 gives $\frac{dU}{ds} = 0$ that is, U is a constant vector. On the other hand,

$$\langle N(s), U \rangle_{G_3} = c\left[\frac{(y''(s))^2 + (z''(s))^2}{\kappa^2(s)} \right] = c$$

and this means that α is a slant helix.

If $a_1 = 0$, we obtain $\langle U, U \rangle_{G_3} = c^2 + a_3^2 = constant.$ Then $a_3 = 0$ and from (4.3) we have $c = 0$. This means that $U = 0$ contradiction.

5. SLANT HELICES IN PSEUDO-GALILEAN SPACE G_3^1

Definition 5.1. A admissible curve α is called a slant helix if there exists a constant vector field U in G_3^1 such that the function $\langle N(s), U \rangle_{G_3^1}$ is constant.

Theorem 5.1. Let α be a admissible curve parameterized by the arc length s in G_3^1.Then α is a slant helix if and only if either one the next two functions

$$\frac{\kappa^2}{\tau^3}\left(\frac{\tau}{\kappa}\right)'. \tag{5.1}$$

is constant everywhere τ does not vanish.

Proof. Let α be a admissible curve in G_3^1. In order to prove Theorem 5.2, we first assume that α is a slant helix. Let U be the vector field such that the function $\langle N(s),U\rangle_{G_3^1} = c\varepsilon$ is constant. There exist smooth functions a_1 and a_3 such that

$$U = a_1(s)T(s) + c\varepsilon N(s) + a_3(s)B(s) \tag{5.2}$$

As U is constant, a differentiation in (5.2) together (5.1) gives

$$\begin{aligned} a_1' &= 0 \\ a_1\kappa + a_3\tau &= 0 \\ a_3' + c\varepsilon\tau &= 0. \end{aligned} \tag{5.3}$$

From the second equation in (5.3) we have

$$a_1 = -a_3\left(\frac{\tau}{\kappa}\right). \tag{5.4}$$

Moreover, if $a_1 \neq 0$,

$$\langle U,U\rangle_{G_3^1} = a_1^2 = constant \tag{5.5}$$

We point out that this constraint, together the second and third equation of (5.3) is equivalent to the very system (5.3). From (5.4) and (5.5), set

$$a_3^2\left(\frac{\tau}{\kappa}\right)^2 = m^2. \tag{5.6}$$

Thus, (5.6) which give

$$a_3 = \pm\frac{m}{\left(\frac{\tau}{\kappa}\right)}$$

on I. The third equation in (5.3) yields

$$\frac{d}{ds}\left(\pm\frac{m}{\left(\frac{\tau}{\kappa}\right)}\right)=-c\varepsilon\tau$$

on I . This can be written as

$$\frac{\kappa^2}{\tau^3}\left(\frac{\tau}{\kappa}\right)'=\pm\frac{c\varepsilon}{m}. \tag{5.7}$$

This shows a part of Theorem 5. 2. Conversely, assume that the condition (5.1) is satisfied. In order to simplify the computations, we assume that the function in (5.1) is a constant, namely, c .We define

$$U=-T+c\varepsilon N+\frac{\tau}{\kappa}B. \tag{5.8}$$

A differentiation of (5.8) together the Frenet equations in G_3^1 gives $\frac{dU}{ds}=0$ that is, U is a constant vector. On the other hand,

$$\langle N(s),U\rangle_{G_3^1}=c\left[\frac{\varepsilon(y''(s))^2-(z''(s))^2}{\kappa^2(s)}\right]=c\varepsilon$$

and this means that α is a slant helix.

If $a_1=0$, we obtain $\langle U,U\rangle_{G_3^1}=c^2-a_3^2=constant.$ Then $a_3=0$ and from (5.3) we have $c=0$. This means that $U=0$ contradiction.

6. REFERENCES

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