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An invariant of regular isotopy for disoriented links

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Abstract: In this paper, we define a two-variable polynomial invariant of regular isotopy, M_K for a disoriented link diagram K. By normalizing the polynomial M_K using complete writhe, we obtain a polynomial invariant of ambient isotopy, N_K, for a disoriented link diagram K. The polynomial N_K is a generalization of the expanded Jones polynomial for disoriented links and is an expansion of the Kauffman polynomial F to the disoriented links. Moreover, the polynomial M_K is an expansion of the Kauffman polynomial L to the disoriented links.

Key words: Disoriented link, disoriented crossing, disoriented regular isotopy, complete writhe, disoriented link polynomial

1. Introduction

We encounter disoriented diagrams in both classical and virtual knot theory. In the classical knot theory, the disoriented link diagrams emerge when calculating the polynomials of oriented links such that the Jones [6, 7] and HOMFLY [5] using an oriented diagram structure of the state summation for the link diagrams. When we split a crossing of an oriented knot diagrams using Kauffman's bracket model [11–13], one of the emerging diagrams is a disoriented one. Moreover, the disoriented diagrams appear when the bracket model is expanded to virtual knots [8, 10] and the arrow polynomials [4] for the virtual knots are calculated and links polynomials are derived from magnetic graphs [14, 15].

Unoriented and oriented link diagrams were considered in the studies in the knot theory until 2018. Altıntaş [1] introduced the theory of disoriented knot in 2018. He defined new concepts such as disoriented crossing, disoriented knot and link and complete writhe. He also extended some basic concepts such as Reidemeister moves, linking number and Kauffman's bracket model [11–13] to disoriented diagrams and generalized the Jones polynomial [6, 7] to disoriented links with the help of the complete writhe.

In [1], a disoriented knot was defined as the embedding of a disoriented circle with two arcs into three dimensional space. In [2], the concept of disoriented knot was redefined by using a circle with $2n$ arcs $(n \in \mathbb{N})$ instead of the circle with two arcs. This new definition of disoriented knot defined as an embedding of a disoriented circle with a $2n$ arcs into $3-$dimensional space or $3-$ dimensional sphere generalize the definition of disoriented knot in [1], which is more advantageous than the definition in [1]. All possible disoriented diagrams of a knot can be drawn using this definition. For example, neither of the last two disoriented diagrams of the

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right-hand trefoil below Definition 2.4 can be defined as the embedding of a disoriented circle with 2 arcs into 3-dimensional space. In contrast, each of the possible disoriented diagrams of the right-hand trefoil is an embedding of a disoriented circle with $2n$ arcs, $n \leq 3$, into 3-dimensional space. Basic diagrammatic methods such as the connected sum of disoriented knots, minimum generating sets of disoriented Reidemeister moves, disorientated Gaussian codes, and disoriented Gaussian diagrams are studied in [2].

In this paper, we define a two-variable Laurent polynomial with integer coefficients and prove that it is a regular isotopy invariant for disoriented links. We denote this polynomial by M_K for a disoriented link diagram K. We prove also that the polynomial N_K obtained by normalizing the polynomial M_K with the help of the complete writhe [1] is an ambient isotopy invariant for the disoriented link K. It can easily be seen that the polynomial M_K is an extension of the Kauffman [9] polynomial L to disoriented link diagrams and N_K is both a generalization of the Jones polynomial [1] for disoriented links and an expansion of the Kauffman polynomial F to the disoriented links.

We plan this paper as follows. The second section contains some of the concepts obtained in [1] and [2], which we will use in the other sections.

In Section 3, we define polynomials M_K and N_K for a disoriented link diagram K and prove that the polynomial N_K is an ambient isotopy invariant for the disoriented link diagrams. We also give some properties of the polynomials M_K and N_K, and prove that the polynomials M_K and N_K are generalizations of uninvariate polynomials for the disoriented links. We give a few examples at the end of the section.

Section 4 contains the proof of the well-definedness and regular isotopy invariance of the polynomial M_K for the disoriented links. Here we define the polynomial M_K inductively and prove that it is a regular isotopy invariant for the disoriented links by using the similar techniques as in [9].

2. Preliminary information

We give some concepts of the disoriented knot theory, which will be used in the next sections.

Definition 2.1 *[2] For each natural number n, let us set $2n$ points on a circle and choose an orientation of each arc between those points such that the consecutive arcs have the reverse orientation. Then the circle is called a disoriented circle.*

Let C be a disoriented circle with $2n$ arcs. Let any arc of C be denoted by A_i and its consecutive arc by B_i. Then C can be represented by a word $A_1B_1A_2B_2...A_nB_n$ such that the orientation of A_i is the reverse of the orientation of B_j for $i,j = 1,2,...,n$ (see Figure 1).

A simple disoriented diagram, disoriented circle with 4 arcs and their replacements were drawn in Figure 1. The fundamental reduction move in Figure 1 is the annihilation of consecutive two cusps on a straightforward noose. This fundamental move allows to delete the reverse oriented arc between two points on it that are in the same local region of the noose. Due to our present disclosure, a disoriented arc can be changed with an oriented arc. In the same way, a disoriented circle can be changed with an oriented circle. For essential information on disoriented configurations, disoriented relations and replacements, see the references [3, 4, 8, 10].

Definition 2.2 *[2] The embedding of a disoriented circle into $3-$dimensional space $\mathbb{R}^3$ (or $3-$dimensional sphere S^3) is called a disoriented knot. The embedding of the disjoint union of k circles into $\mathbb{R}^3$ is called a disoriented link of k-components, where at least one of the circles is disoriented.*

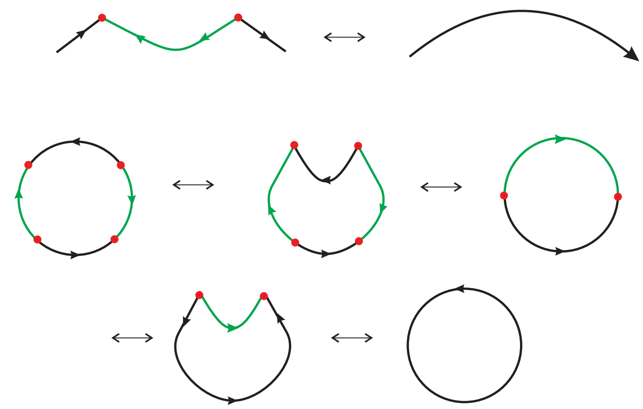

Figure 1. Elementary disoriented diagrams and replacements.

Definition 2.3 *[2] Let K be a disoriented knot. A crossing of K is called disoriented if its underpass and overpass arcs have inverse orientations. Namely, let K be an embedding of a disoriented circle C. If A_i and B_j are the arcs of C, one of the overpass and underpass arcs is A_i and the other B_j. A crossing of K is called oriented if it is not disoriented. An oriented knot is a disoriented knot with zero disoriented crossing (see Figure 2).*

Definition 2.4 *[2] Let the components of a two-component links L ring be denoted by K_1 and K_2. Let us select a disorientation of both K_1 and K_2 and denote two arcs of K_1 by A_i^1 and B_i^1 and two arcs of K_2 by A_i^2 and B_i^2. Then, if one of the following holds, a crossing of L is disoriented:*

1. *One of the overpass and underpass arcs of the crossings is A_i^1, and the other is A_i^2 or B_i^2.*
2. *One of the overpass and underpass arcs of the crossings is B_i^1, and the other is A_i^2 or B_i^2.*

   Or else, the crossing is called oriented.

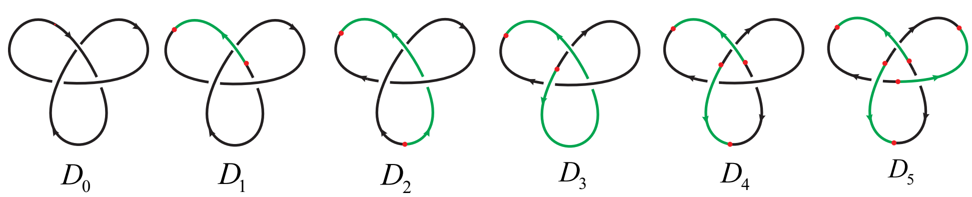

Figure 2. Oriented and disoriented diagrams of the right-hand trefoil.

In Figure 2, we draw the possible disoriented diagrams of the right-hand trefoil. Note that these diagrams are embeddings of a disoriented circle C with $2n$ arcs, $n \leq 3$,$n \in \mathbb{N}$. The diagram D_0 has no disoriented crossing. Therefore, it is an embedding of C such that only one arc of C is crossed with itself. The diagrams D_1, D_2, and D_3 are embeddings of C such that its two opposite arcs are crossed with each other. The diagram D_4 is an embedding of C such that two opposite arcs of its four arcs are crossed with each other and the other two opposite arcs are crossed with each other. The diagram D_5 is an embedding of C such that every two consecutive opposite arcs of its six arcs are crossed with each other.

Observation 2.1 [2] *A disoriented knot with n crossings is an embedding of a disoriented circle with a maximum of $2n$ arcs.*

To define the connected sum of disoriented knots, we denote a disoriented knot K with n crossings in S^3 by the pair (S^3, K). Suppose that A^i and B^j are the arcs of a disoriented circle C of which K is embedding. Let P be a point on K that is different from crossing points of K. Then P either belongs to arc A^i or B^j or is an intersection point of A^i and B^j, $i, j \in \{1, 2, ..., n\}$. Note that if P is an intersection point of A^i and B^j, then $P \in A^i \cap B^i$ or $P \in A^i \cap B^{i-1}$ or $P \in A^i \cap B^{i+1}$ or $P \in A^1 \cap B^n$.

Definition 2.5 [2] *Let (S^3, K_1) and (S^3, K_2) be two disoriented knots, A_k^i and B_k^j be arcs of the disoriented circles C_k which K_k are embeddings, $k \in \{1, 2\}$, $i, j \in \{1, 2, ..., n\}$. Let P_k be a point on K_k that is no crossing point. The connected sum of the disoriented knots K_1 and K_2 is a disoriented knot obtained from the disjoint union of the manifold pairs $(S^3 - intV_k^3,\ K_k - intV_k^1)$, $(k = 1, 2)$, by pasting their boundaries along a disorientation reserving homeomorphism $\varphi : (\partial U_2^3,\ \partial U_2^1) \to (\partial U_1^3,\ \partial U_1^1)$, where U_k^3 is a 3-ball with the center P_k and U_k^1 is a 1-ball with the center P_k. The connected sum of K_1 and K_2 is denoted by $K_1 \# K_2$.*

Note that $K_1 \# K_2$ is independent of the points P_k. Therefore, $K_1 \# K_2$ is uniquely determined by K_1 and K_2.

The structure can be defined as follows: $K_1 \# K_2$ is a disoriented knot formed by connecting any diagram of K_1 with that of K_2 in Figure 3.

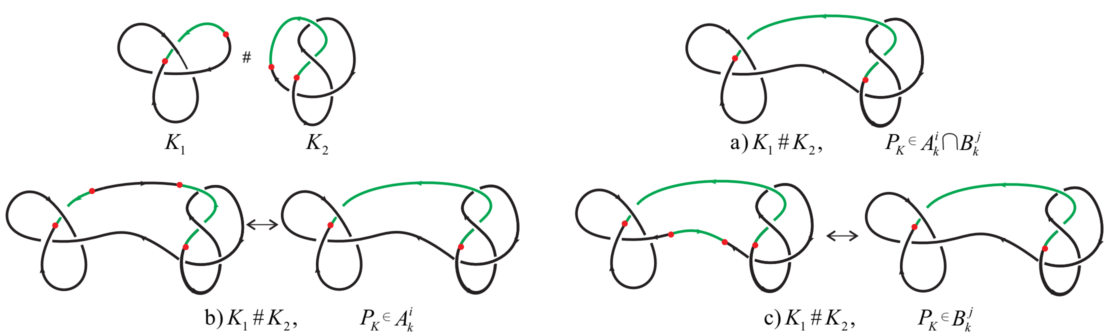

Figure 3. The connected sum of two disoriented knots.

In [2], the Reidemeister moves for disoriented diagrams are given as a generalization of the Reidemeister moves of the oriented diagrams. For collections of oriented Reidemeister moves, see Polyak [16]. Polyak proves that the set containing Reidemeister moves $\Omega 1a$, $\Omega 1b$ in Figure 4, $\Omega 2a$ in Figure 5, and $\Omega 3a$ in Figure 6 generate all oriented Reidemeister moves. This generating set of Reidemeister moves has the minimum number of generators.

To create generating sets of disoriented Reidemeister moves, we need to expand the moves in the generating sets of oriented Reidemeister moves to disoriented diagrams. We illustrate these moves in Figures 4–6. The moves $\Omega 0a$ and $\Omega 0b$ in Figure 4 are planar moves on disoriented diagrams.

In Figure 5, the move $\Omega 2e$ is a disoriented expansion of the moves $\Omega 2a$ and $\Omega 2c$. The move $\Omega 2f$ is a disoriented expansion of the moves $\Omega 2a$ and $\Omega 2b$. The move $\Omega 2g$ is a disoriented expansion of the move $\Omega 2b$. The move $\Omega 2h$ is a disoriented expansion of the move $\Omega 2c$. The move $\Omega 2i$ is also a disoriented expansion of the moves $\Omega 2a$, $\Omega 2b$, and $\Omega 2c$.

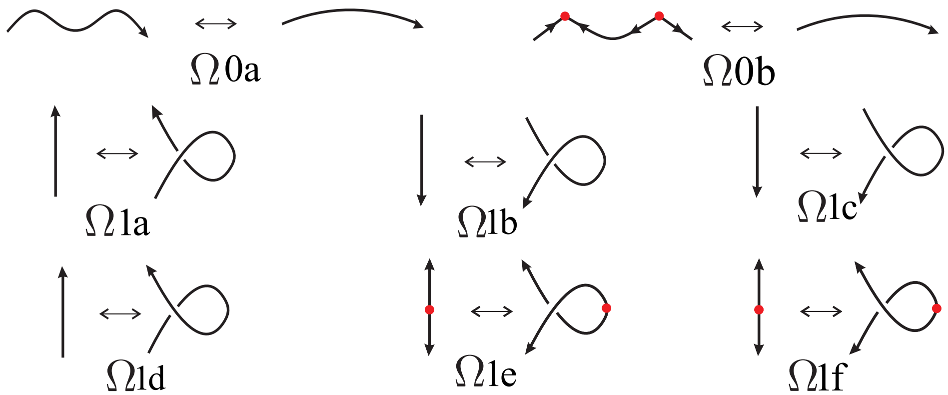

Figure 4. Planar and some disoriented Reidemeister moves of type I.

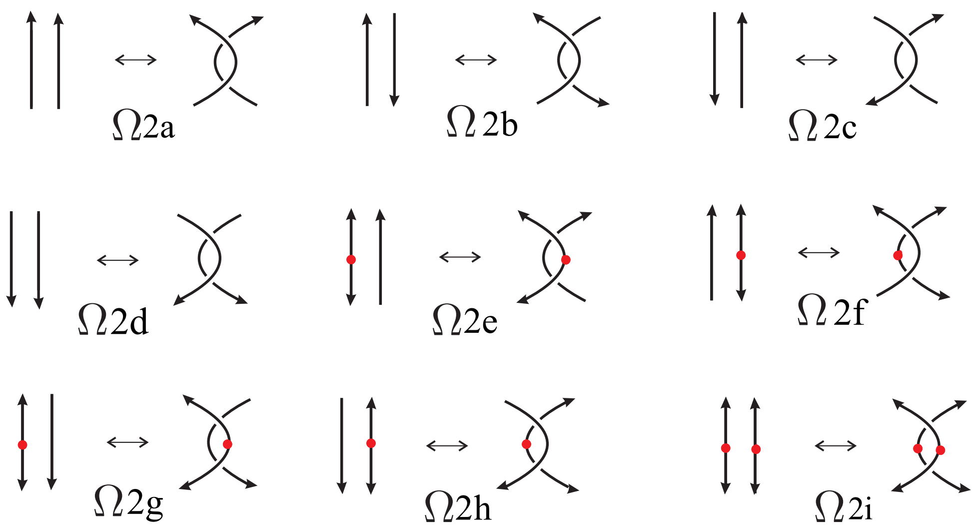

Figure 5. Some disoriented Reidemeister moves of type II.

Definition 2.6 *[2] The equivalence relation created by the moves* $\Omega 2$ *and* $\Omega 3$ *(and the planar moves) is called regular isotopy and the equivalence relation created by the* $\Omega 1$*,* $\Omega 2$*, and* $\Omega 3$ *is called ambient isotopy on disoriented diagrams.*
The generating set $S = \{\Omega 1a,\ \Omega 1b,\ \Omega 1e,\ \Omega 1f,\ \Omega 2a,\ \Omega 2e,\ \Omega 2f,\ \Omega 2i,\ \Omega 3a_i : i \in \{0, ..., 7\}\}$ *of disoriented moves has the minimal numbers of generators. If* D *and* D' *are two disoriented diagrams of the same disoriented link, then we can pass from* D *to* D' *by planar moves and a sequence of disoriented moves in the generating set* S*.*

Definition 2.7 *[1] Suppose* D *is a disoriented regular diagram of a knot (or link)* K*. The complete writhe of* D *is denoted by* $cw(D)$ *an is defined by equation*

$$cw(D) = \sum_{o} \varepsilon(o) - \sum_{d} \varepsilon(d).$$

In this equation, the first sum runs over the oriented crossings of D *and latter over the disoriented crossings of*

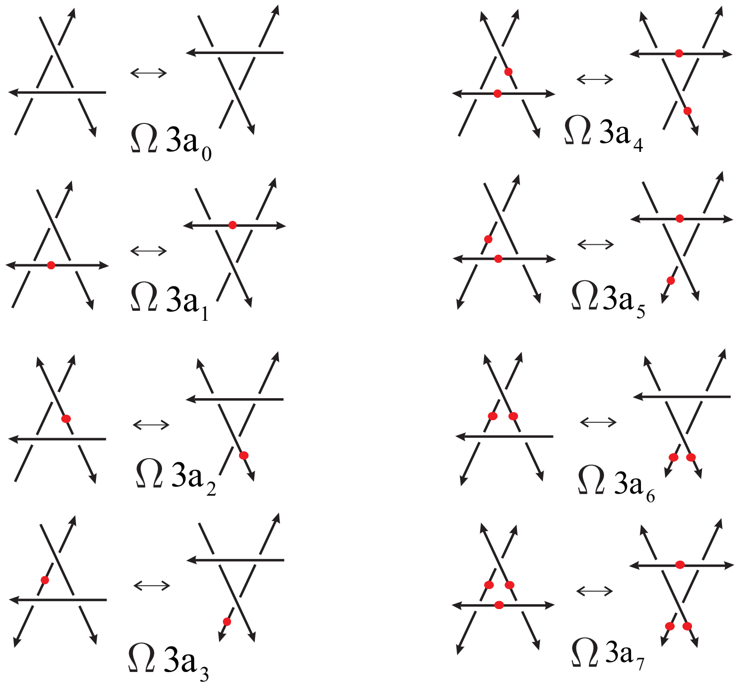

Figure 6. Some disoriented Reidemeister moves of type III.

D, and $\varepsilon(o)$ denotes the sign of an oriented crossing of D and $\varepsilon(d)$ the sign of a disoriented crossing of D.

$cw(D)$ is an invariant of regular isotopy for the disoriented diagram D and the complete writhes of all the disoriented diagrams of a nontrivial link are equal [1].

The bracket expansion for oriented link diagrams can be adapted as an oriented bracket state model [4, 8]:

$$\begin{aligned} &\langle K_+\rangle = A\langle K_0\rangle + A^{-1}\langle K_\infty\rangle, \\ &\langle K_-\rangle = A^{-1}\langle K_0\rangle + A\langle K_\infty\rangle, \\ &\delta = -A^2 - A^{-2}, \quad \langle \bigcirc \sqcup D\rangle = \delta\langle D\rangle, \end{aligned} \tag{2.1}$$

where K_+, K_-, K_0, and K_∞ are diagrams in Figure 7, $\bigcirc$ is an oriented diagram with zero-crossing of unknot and D an oriented link diagram and $\sqcup$ is disjoint union.

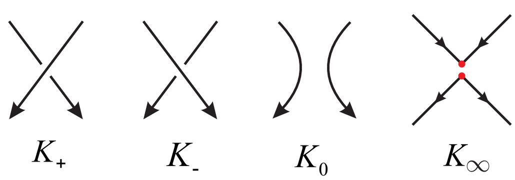

Figure 7. Crossings and smoothings.

We also use the model (2.1) for disoriented link diagrams and call the expanded bracket polynomial for disoriented links [1].

Lemma 2.8 [1] *$\langle I\rangle = (-A^3)\langle I_0\rangle$ and $\langle I'\rangle = (-A^3)\langle I_1\rangle$, where I, I_0, I', and I_1 are diagrams in Figure 8.*

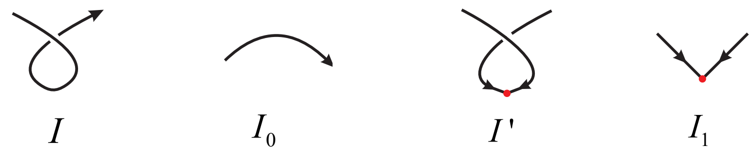

Figure 8. Some Reidemeister moves of type I.

Definition 2.9 *[1] Let us assume that $\langle D\rangle$ is the bracket polynomial of a D disoriented diagram of a link K and $cw(D)$ is its complete writhe number. The polynomial $\mathcal{T}_K \in \mathbb{Z}[A, A^{-1}]$ defined by the formula*

$$\mathcal{T}_K(A) = (-A^3)^{-cw(D)}\langle D\rangle$$

is called the complete normalized polynomial.

The complete normalized polynomial is a ambient isotopy invariant for the disoriented link diagrams [1].

3. Polynomial invariants for disoriented links

In this section, we define a two-variable polynomial invariant of regular isotopy, $M_K(a,x)$, for disoriented link diagrams that generalizes the extended bracket polynomial. By normalizing the polynomial M_K with complete writhe, we obtain a polynomial invariant of ambient isotopy $N_K(a,x)$ for disoriented links. The polynomial N_K generalizes extended Jones polynomial. The polynomials M_K and N_K are the extensions of Kauffman [9] polynomials L and F for the disoriented link diagrams, respectively.

Definition 3.1 *Let K be a disoriented link diagram and $M_K \in \mathbb{Z}[a, a^{-1}, x, x^{-1}]$ be a Laurent polynomial in the variables a, x appointed to the disoriented link diagram L. The polynomial M_K meets the axioms:*

1. *If K_1 and K_2 are regularly isotopic link diagrams, then $M_{K_1} = M_{K_2}$,*
2. *$M_O = 1$,*
3. *$M_{I_+} = aM_{I_0}, \quad M_{I_-} = a^{-1}M_{I_0}$,*
4. *$M_{I'_+} = a^{-1}M_{I'_0}, \quad M_{I'_-} = aM_{I'_0}$,*
5. *$M_{K_+} + M_{K_-} = x(M_{K_0} + M_{K_\infty})$,*

where K_+, K_-, K_0, K_∞, $I_+, I_-, I_0, I'_+, I'_-$, and I'_0 are diagrams given in Figure 9, O is the unknot with zero-crossing.

Theorem 3.1 *The polynomial M_K is a well-defined polynomial of regular isotopy for the disoriented link diagram K.*

We will prove this theorem in the next section.

Definition 3.2 *We define a polynomial $N_K \in \mathbb{Z}[a, a^{-1}, x, x^{-1}]$ for a disoriented link diagram K by the equality*

$$N_K = a^{-cw(K)}M_K.$$

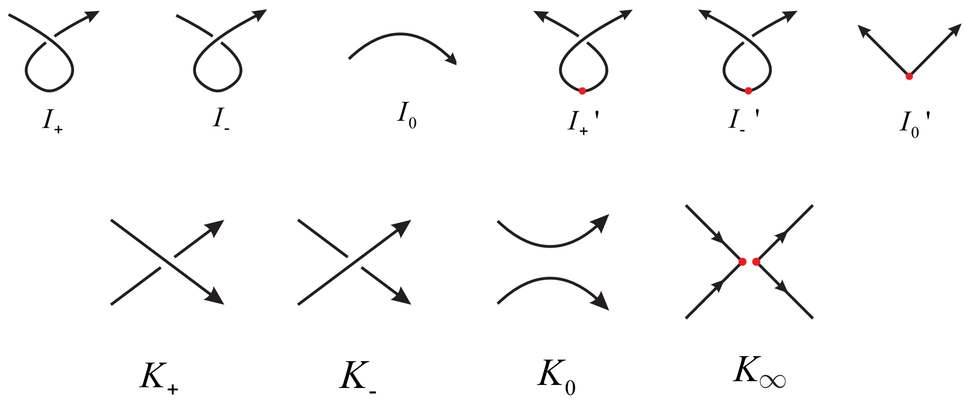

Figure 9. Crossings and smoothings.

Theorem 3.2 *The polynomial* N_K *is an ambient isotopy invariant for the disoriented link diagram* K.

Proof Since $cw(K)$ is an invariant of regular isotopy, $a^{-cw(K)}$ is also an invariant of regular isotopy. Hence, N_K is an invariant of regular isotopy. It is then sufficient to check the behavior of N_K under the disoriented move of type I. Since $cw(I_+) = 1 + cw(I_0)$, $cw(I_-) = -1 + cw(I_0)$, $cw(I'_+) = -1 + cw(I'_0)$ and $cw(I'_-) = 1 + cw(I'_0)$, we have

$$N_{I_+} = a^{-cw(I_+)}M_{I_+} = a^{-(1+cw(I_0))}aM_{I_0} = a^{-cw(I_0)}M_{I_0} = N_{I_0},$$

$$N_{I_-} = a^{-cw(I_-)}M_{I_-} = a^{-(-1+cw(I_0))}a^{-1}M_{I_0} = a^{-cw(I_0)}M_{I_0} = N_{I_0},$$

$$N_{I'_+} = a^{-cw(I'_+)}M_{I'_+} = a^{-(-1+cw(I'_0))}a^{-1}M_{I'_0} = a^{-cw(I'_0)}M_{I'_0} = N_{I'_0},$$

$$N_{I'_-} = a^{-cw(I'_-)}M_{I'_-} = a^{-(-1+cw(I'_0))}aM_{I'_0} = a^{-cw(I'_0)}M_{I'_0} = N_{I'_0},$$

where $I_+, I_-, I_0, I'_+, I'_-$, and I'_0 are diagrams in Figure 9. These diagrams correspond to disoriented Reidemeister moves of type I drawn in Figure 4 □

Theorem 3.3 *Let* K *be a disoriented diagram. Then*

$$\langle K\rangle(A) = M_K(-A^3, A + A^{-1}),$$

$$\mathcal{T}_K(A) = N_K(-A^3, A + A^{-1}).$$

Proof Let us just show that $\langle K\rangle(A) = M_K(-A^3, A + A^{-1})$. Others are shown similarly. From the bracket models,

$$\langle K_+\rangle = A\langle K_0\rangle + A^{-1}\langle K_\infty\rangle,$$

$$\langle K_-\rangle = A^{-1}\langle K_0\rangle + A\langle K_\infty\rangle,$$

we get $\langle K_+\rangle + \langle K_-\rangle = (A + A^{-1})(\langle K_0\rangle + \langle K_\infty\rangle)$. This is a special case of the polynomial M_K by $x = A + A^{-1}$. It is clear that the other axioms are satisfied by taking $a = -A^3$. □

Proposition 3.3 *Let K^* be the mirror image of a disoriented link diagram K. Then,*

$$M_{K^*}(a, x) = M_K(a^{-1}, x),$$
$$N_{K^*}(a, x) = N_K(a^{-1}, x).$$

Proof Since K^* is obtained from K by reversing all crossings, it is obvious that $cw(K^*) = -cw(K)$. Moreover, this appears by replacement of a by a^{-1} in the axioms 3 and 4 of Definition 3.1. Thus, a calculation of M_{K^*} results in an identical calculation of M_K with a replaced by a^{-1}. Therefore, $M_{K^*}(a, x) = M_K(a^{-1}, x)$. Similarly, $N_{K^*}(a, x) = N_K(a^{-1}, x)$. □

Remark 3.4 *As a consequence of Proposition 3.3, if $N_K(a, x) \neq N_K(a^{-1}, x)$, K is not ambient isotopic to its mirror image.*

Example 3.5 *Let us calculate the polynomials M and N of the disoriented diagrams in Figure 10. From the definitions 3.1 and 3.2, we have*

$$M_{K_1} = aM_\circ = a, \qquad N_{K_1} = a^{-cw(K_1)}M_{K_1} = 1,$$
$$M_{K_1^*} = a^{-1}M_\circ = a^{-1}, \qquad N_{K_1^*} = a^{-cw(K_1)}M_{K_1^*} = 1,$$
$$M_{K_2} = a^{-1}M_\circ = a^{-1}, \qquad N_{K_2} = a^{-cw(K_2)}M_{K_2} = 1,$$
$$M_{K_2^*} = aM_\circ = a, \qquad N_{K_2^*} = a^{-cw(K_2)}M_{K_2^*} = 1.$$

By the relation $M_{K_+} + M_{K_-} = x(M_{K_0} + M_{K_\infty})$, we have

$$M_{K_1} + M_{K_1^*} = x(M_{\circ\circ} + M_\circ)$$
$$aM_\circ + a^{-1}M_\circ - xM_\circ = xM_{\circ\circ}$$
$$a + a^{-1} - x = xM_{\circ\circ} \qquad (\textit{with} \quad M_{\circ\circ} = \delta M_\circ)$$
$$\delta = (a + a^{-1})x^{-1} - 1.$$

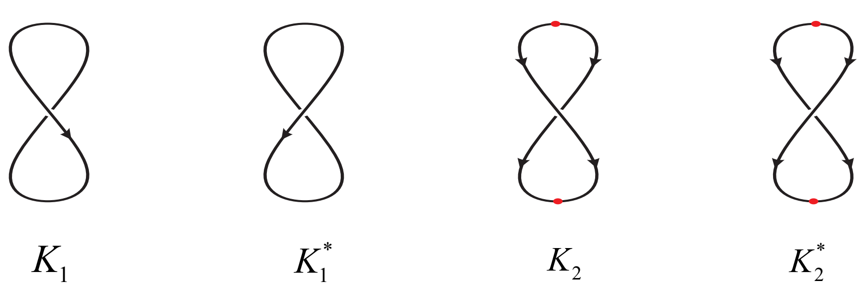

Figure 10. The disoriented unknots with one crossing.

Example 3.6 *Let* L *be a disoriented link in Figure 11. Then,*

$$M_L + \delta = x(a + a^{-1})$$
$$M_L = x(a + a^{-1}) - x^{-1}(a + a^{-1}) + 1$$
$$M_L = (a + a^{-1})(x - x^{-1}) + 1$$

and

$$N_L = a^{-cw(L)}M_L = a^{-2}[(a + a^{-1})(x - x^{-1}) + 1]$$
$$N_L = (a^{-1} + a^{-3})(x - x^{-1}) + a^{-2}.$$

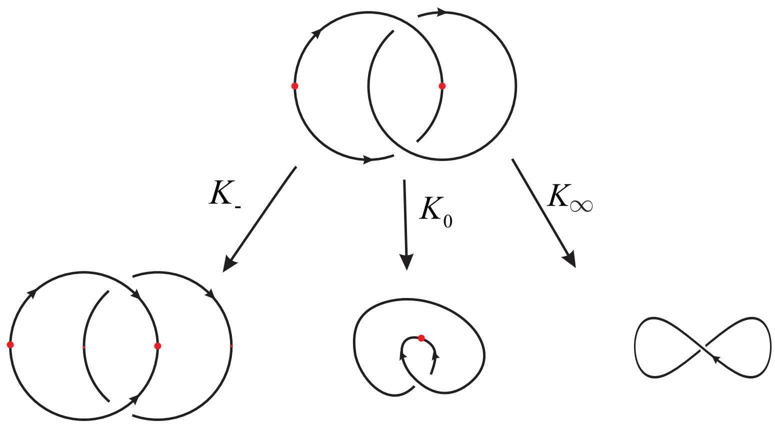

Figure 11. A disoriented link.

Example 3.7 *If* K *is a disoriented diagram of the trefoil knot in Figure 12, then*

$$M_K + M_{K_1} = x(M_{K'} + M_L)$$
$$M_K = x[a^{-2} + (a + a^{-1})(x - x^{-1} + 1)] - a$$
$$M_K = (-2a - a^{-1}) + (1 + a^{-2})x + (a + a^{-1})x^2,$$

where $M_{K'} + M_\circ = z(M_{K_2} + M_{K_2^*}) \Rightarrow M_{K'} = a^{-2}$. *Since*

$$cw(K) = \sum_o \varepsilon(o) - \sum_d \varepsilon(d) = 2 - (-1) = 3,$$

$$N_K = a^{-cw(K)}M_K$$
$$N_K = (-2a^{-2} - a^{-4}) + (a^{-3} + a^{-5})x + (a^{-2} + a^{-4})x^2.$$

Result 3.4 *As a consequence of Example 3.5, it is clear that for a disoriented knot diagram* K, $M_{\circ\sqcup K} = \delta M_K$, $N_{\circ\sqcup K} = \delta N_K$. *Also for any disoriented knot diagrams* K_1 *and* K_2, $M_{K_1\sqcup K_2} = \delta M_{K_1}M_{K_2}$ *and* $N_{K_1\sqcup K_2} = \delta N_{K_1}N_{K_2}$.

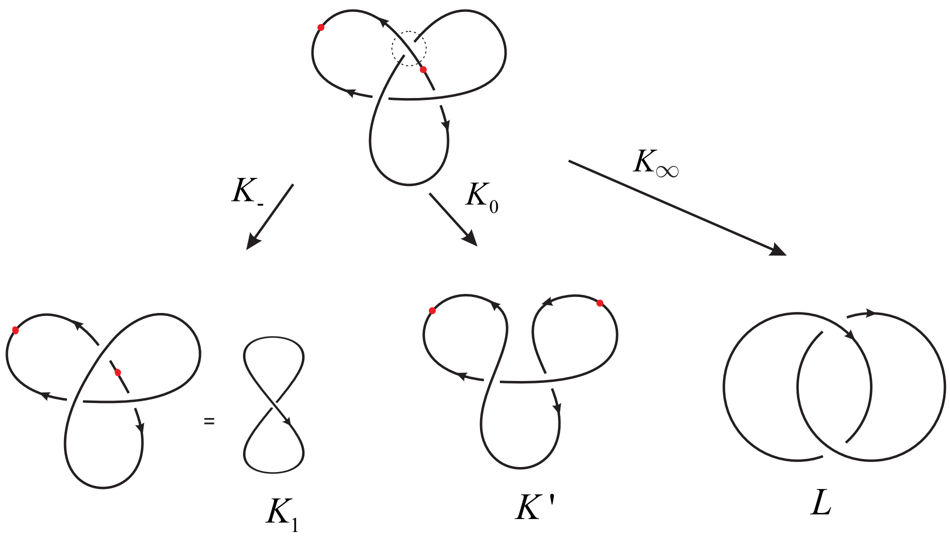

Figure 12. A smoothing of disoriented trefoil knot.

Proposition 3.8 *Let $K = K_1 \# K_2$ be the connected sum of two disoriented knot diagrams K_1 and K_2. Then,*

$$M_{K_1\#K_2} = M_{K_1}M_{K_2}, \tag{3.1}$$

$$N_{K_1\#K_2} = N_{K_1}N_{K_2}. \tag{3.2}$$

Proof It is sufficient to prove that (3.1) is true. The equation (3.2) can be shown in a similar way. If the diagram K_1 (or K_2) in $K_1\#K_2$ is inverted according to right-hand orientation, $K_1^+\#K_2$ is obtained. If the diagram K_1 (or K_2) in $K_1\#K_2$ is inverted according to left-hand orientation, $K_1^-\#K_2$ is obtained. Moreover, note that the diagram $K_1\#K_2$ is derived from $K_1 \sqcup K_2$, see Figure 13.

If K_1 has $n-$crossings, the diagrams K_1^+ and K_1^- has $n+1-$crossings. Thus, from the relation

$$M_{K_+} + M_{K_-} = x(M_{K_0} + M_{K_\infty})$$

we have

$$\begin{aligned} M_{K_1^+\#K_2} + M_{K_1^-\#K_2} &= x(M_{K_1\sqcup K_2} + M_{K_1\#K_2}) \\ aM_{K_1\#K_2} + a^{-1}M_{K_1\#K_2} &= x\delta M_{K_1}M_{K_2} + xM_{K_1\#K_2} \\ [(a+a^{-1})x^{-1} - 1]M_{K_1\#K_2} &= \delta M_{K_1}M_{K_2} \\ M_{K_1\#K_2} &= M_{K_1}M_{K_2}. \end{aligned}$$

□

4. Well-definedness and invariance of the polynomial M

In this section, we define the polynomial M inductively similar to the Kauffman's inductive definition [9] for disoriented links. For this, it is necessary to switchings and eliminations of the disoriented crossings. Here we denote by T_iK for the disoriented link acquired by switching the disoriented link K at any ith crossing, and

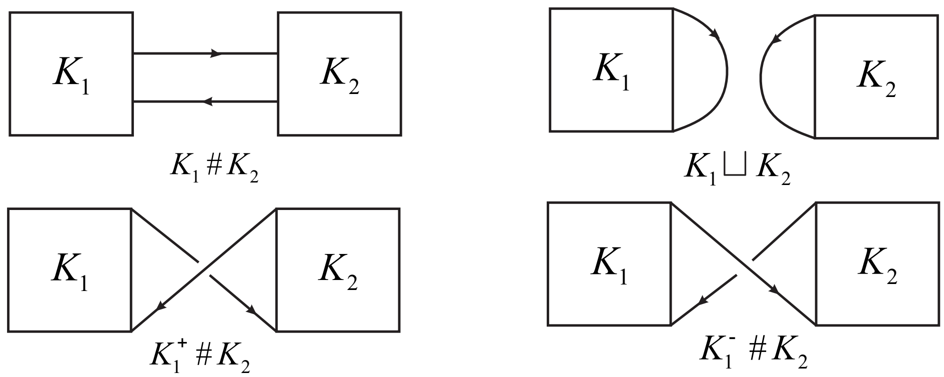

Figure 13. Connected sum.

E_iK, F_iK for the oriented and disoriented splicings at the ith crossing, respectively, see Figure 14. We want to give a definition to M_K such that the identity

$$M_K + M_{T_iK} = x(M_{E_iK} + M_{F_iK})$$

is a consequence of the definition. The motivation for this definition we have adopted is demonstrated by the following remarks. Definition 3.1 will follow these remarks.

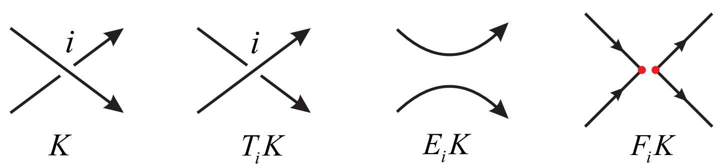

Figure 14. Smoothing and elimination of the ith crossing.

Definition 4.1 *(Inductive definition) Assume that K is a disoriented knot diagram of $n+1-$ crossings. Label each crossing with $0, 1, ..., n$. Then, the following list of equations can be written*

$$M_K + M_{T_0K} = x(M_{E_0K} + M_{F_0K}),$$

$$M_{T_0K} + M_{T_1T_0K} = x(M_{E_1T_0K} + M_{F_1T_0K}),$$

$$\vdots$$

$$M_{T_{n-1}\cdots T_0K} + M_{T_n\cdots T_0K} = x(M_{E_nT_{n-1}\cdots T_0K} + M_{F_nT_{n-1}\cdots T_0K}).$$

We denote the result of switching all crossings by $\hat{K} = T_n \cdots T_0K$ and elimination operators by $A_iK = E_iT_{i-1}\cdots T_0K$, $B_iK = F_iT_{i-1}\cdots T_0K$. Then, by successive addition and subtract of the above equations, we can show that

$$M_K = (-1)^{n+1}M_{\hat{K}} + x(\sum_{i=0}^{n}(-1)^i(M_{A_iK} + M_{B_iK})). \tag{4.1}$$

The formula (4.1) gives how to compute M_K and the results of K implemented to smaller disoriented link diagrams. We choose a switching sequence of K. Then if K is a disoriented knot, $\hat{K}$ is an unknot. If K is a disoriented link, $\hat{K}$ is a split disoriented link. In calculating disoriented links, we have the precept

$$M_{K_1 \sqcup K_2} = \delta M_{K_1} M_{K_2}, \tag{4.2}$$

where $\delta = (a + a^{-1})z^{-1} - 1$ as in Section 3. The best way to describe an inductive definition is to use a normal unknot connected with a disoriented knot diagram with directed base-point. The normal unknot is built as follows: Assume that K is a disoriented knot diagram, U is its planar shade and p is a point an arc of U. We draw a disoriented knot diagram $\hat{K} = \hat{K}(U, p)$ by moving along U in the direction p and doing overpass the crossing on the first pass at each crossing. This reveals a disoriented unknotted diagram as in Figure 15.

The normal unknot $\hat{K} = \hat{K}(U, p)$ is used to reveal a special unknotting sequence for the disoriented knot diagram K. We move K from p and tick each crossing that differs from the corresponding crossing in $\hat{K}$. We tag the ticked crossing with $n, n-1, \cdots, 0$ in descending order from base-point. Therefore, by switching these crossings $\hat{K}$ acquired from K and we obtain $\hat{K} = T_n T_{n-1} \cdots T_0 K$. This switching sequence is specified by the choice of directed base-point on K. Hence, the polynomial M on normal unknots is defined by the equal

$$M_{\hat{K}(U,p)} = a^{cw(\hat{K}(U,p))}. \tag{4.3}$$

In order to take advantage of formula (4.2), it is also necessary to decompose the components with a switching sequence. the formula (4.1) can be related to a split disoriented link rather than a disoriented unknot. Now, we have a procedure of recursive calculation using the formulas (4.1), (4.2), and (4.3) such that the calculations finally depend only on the values of M at normal unknots. In order to formalize these processings to obtain an inductive definition, it is helpful to make up a notation for the second side of equality (4.1).

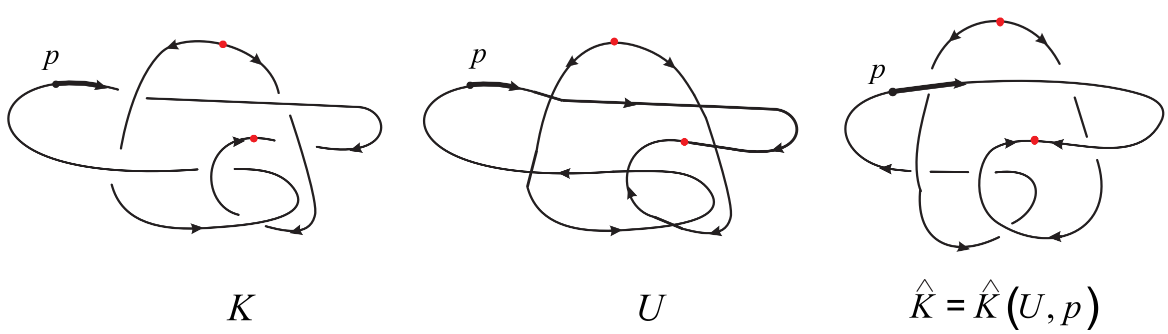

Figure 15. Normal unknot.

Definition 4.2 *Let K be a disoriented link diagram and $\alpha = (\alpha_n, \alpha_{n-1}, \cdots, \alpha_0)$ be an ordered sequence of labels for crossing of K. Let A_i^α and B_i^α be the operators given by formulas $A_i^\alpha = E_i T_{\alpha_i} T_{\alpha_{i-1}} \cdots T_{\alpha_0} K$ and $B_i^\alpha = F_i T_{\alpha_i} T_{\alpha_{i-1}} \cdots T_{\alpha_0} K$. Let $\hat{K}(\alpha) = T_{\alpha_n} T_{\alpha_{n-1}} \cdots T_{\alpha_0} K$, $\sum_K(\alpha) = \sum_{i=0}^{n} (-1)(M_{A_i^\alpha K} + M_{B_i^\alpha K})$ and $\psi_K(\alpha) = (-1)^{|\alpha|+1} M_{\hat{K}(\alpha)} + x \sum_K(\alpha)$, where $|\alpha| = n$. Note that we want that $\psi_K(\alpha) = M_K$. $\psi_{K(\alpha)}$ will be used for logical aims.*

We now give the inductive definition of M_K.

Definition 4.3 *Let $K = K_1 \cup K_2 \cup K_2 \cup \cdots \cup K_n$ be a disoriented link of n components. $K - K_i$ denotes the disoriented link ejecting the ith component from K. We assume that K_i represent disoriented knot diagram obtained from K by wiping every the components $K_1, K_2, \cdots, K_{i-1}, K_{i+1}, \cdots, K_n$.*

1. *If $\hat{K} = K(U, p)$ is a normal unknot, then $M_{\hat{K}} = a^{cw(\hat{K})}$.*
2. *If K_1 is a disoriented knot overlaying a disoriented link diagram K_2, then $M_{K_1 \sqcup K_2} = \delta M_{K_1} M_{K_2}$ where $\delta = (a + a^{-1})x^{-1} - 1$.*
3. *Let $K = K_1 \cup K_2 \cup K_2 \cup \cdots \cup K_n$ be a disoriented link diagram.*
   a. *If a component overlies the others, part (2) is applied.*
   b. *Let no component K_i overlie the others. Assume that $p_1, \cdots p_n$ are directed base-points on $K_1, \cdots, K_n$, $\bar{p}_1, \cdots, \bar{p}_n$ are the same base-points endowed with the reversed direction, $\alpha(P_i)$ is sequence of undercrossings of K_i with $K - K_i$ such that $\hat{K}(\alpha(p_i)) = K \sqcup (K - K_i)$ with K_i overcrossing the remainder of these components. Since p_i determines $\alpha(p_i)$, $\sum_K (\alpha(p_i)$ depends only on the choice of directed base-point p_i. Then, we define M_K by the formula*

$$M_K = \frac{1}{2n}[\sum_{i=1}^{|\alpha(p_i)|} (-1)^{|\alpha(p_i)|+1} \delta M_{K_i} M_{(K-K_i)} + x \sum_K \alpha(p_i)$$
$$+ \sum_{i=1}^{|\alpha(\bar{p}_i)|} (-1)^{|\alpha(\bar{p}_i|+1)} \delta M_{K_i} M_{(K-K_i)} + x \sum_K \alpha(\bar{p}_i)].$$

4. *Assume K is a disoriented knot diagram, p is a directed base-point for K, $\bar{p}$ is the same base-point with reversed direction, and $\alpha(p)$ and $\alpha(\bar{p})$ are the switching sequences determined by p and $\bar{p}$, respectively. Then, we can define M_K by the formula*

$$M_K = \frac{1}{2}[(-1)^{|\alpha(p)|+1} M_{\hat{K}(\alpha(p))} + x \sum_K \alpha(p) + (-1)^{|\alpha(\bar{p})|+1} M_{\hat{K}(\alpha(\bar{p}))} + x \sum_K \alpha(\bar{p})].$$

Thus, the inductive definition of M_K is complete.

Since we include summations at both of the associated orientations for each base-point, it is sufficient to prove inductively that the definitions do not depend on the choice of base-point. Entire induction confirmations will be established on the number of crossings of the disoriented link diagrams. Hence, in every case, we will assume that it is verified that M_K has a certain property for all diagrams with less than n crossings. We prove that Definition 4.3 results in this property for disoriented links with n crossings.

Definition 4.4 *The inductive hypothesis of M_K defined in Definition 4.3 is as follows:*

1. *M_K is independent of base-point (well defined) on disoriented link diagrams with less than n crossings.*

2. M_K meets the axioms:

$$M_{I_+} = aM_{I_0}, \qquad M_{I_-} = a^{-1}M_{I_0},$$
$$M_{I'_+} = a^{-1}M_{I'_0}, \qquad M_{I'_-} = aM_{I'_0},$$
$$M_K + M_{T_iK} = x(M_{E_iK} + M_{F_iK}),$$

where K has $< n$ crossings (and I_+, I_-, I'_+, I'_- have $< n$ crossings.)

3. M_K is invariant underneath the disoriented Reidemeister moves of type II and type III that do not rise the number of crossings of disoriented diagram. Namely, if K has $< n$ crossings and K' is acquired from K by the disoriented Reidemeister moves of type II and III that do not rise the number of crossings, then $M_K = M_{K'}$.

To prove that M_K is well-defined, it is necessary to show it with respect to the base-point in 4.3 (3) and 4.3 (4). The next lemma concerns 4.3 (3)

Lemma 4.5 *Assume that $\alpha = (\alpha_n, \alpha_{n-1}, \cdots, \alpha_0)$ is a choice of tags for a subset of different crossings of a disoriented link K and $\beta = (\alpha_0, \alpha_n, \alpha_{n-1} \cdots, \alpha_1)$. Then, $\sum_K \alpha = \sum_K \beta$ is defined as in Definition 4.2. That is, $\sum_K \alpha$ is invariant under cyclic permutation of α.*

Proof The proof is similar to that of Lemma 6.6 in [9]. □

Remark 4.6 *It is obvious from Lemma 4.5 that the formula of M_K given in Definition 4.3 (3) is independent of the choice of base-point. Thus, it remains to show independence from the base-point in case Definition 4.3 (4).*

Lemma 4.7 *We consider the two roads of splicing a normal unknot at the first crossing past to a directed base-point. In one of the roads, the splice uncovers an unknot and in the other it uncovers a disoriented unlink constituted of two normal unknots with one overlying the other.*

Proof Proof follows from the definition of normal unknot. We think the first crossing past the base-point. Starting at the base-point and advancing in the direction it pointed, we advance over the crossing i. The diagram drawn afterwards lays over the remainder of the unknot diagram.

At the crossing i, one of the oriented and disoriented separations cause a disoriented unlink and the other a connected sum of two unknots, see Figure 16. This disoriented unknot diagram is not normal.

As seen in Figure 16, the crossing i is the first crossing encountered over advancing from the base-point of the normal unknot $\hat{K}$. Here there are two splices $E_i\hat{K}$ and $F_i\hat{K}$. $E_i\hat{K}$ is an unlink with two normal unknots, while $F_i\hat{K}$ is an unknot disoriented diagram. If we say $E_i\hat{K} = K_1 \sqcup K_2$, where K_1 and K_2 in this link diagram. Then, $F_i\hat{K} = K_1 \# K_2$. It can be easily seen that the normal unknot corresponding to $F_i\hat{K}$ is $K_1^* \# K_2$ where K_1^* is the mirror image of K_1. A fact regarding normal unknot diagrams generalizing to diagram $F_i\hat{K}$ is that normal unknot diagrams are either constituted completely of curls I_+, I_-, I'_+, and I'_- or they are simplified by the disoriented Reidemeister moves of type II and III. Consequently, we have $M_{F_i\hat{K}} = a^{cw(F_i\hat{K})}$ by applying Definition 4.3 (3) and that the complete writhe is regular isotopy invariance. □

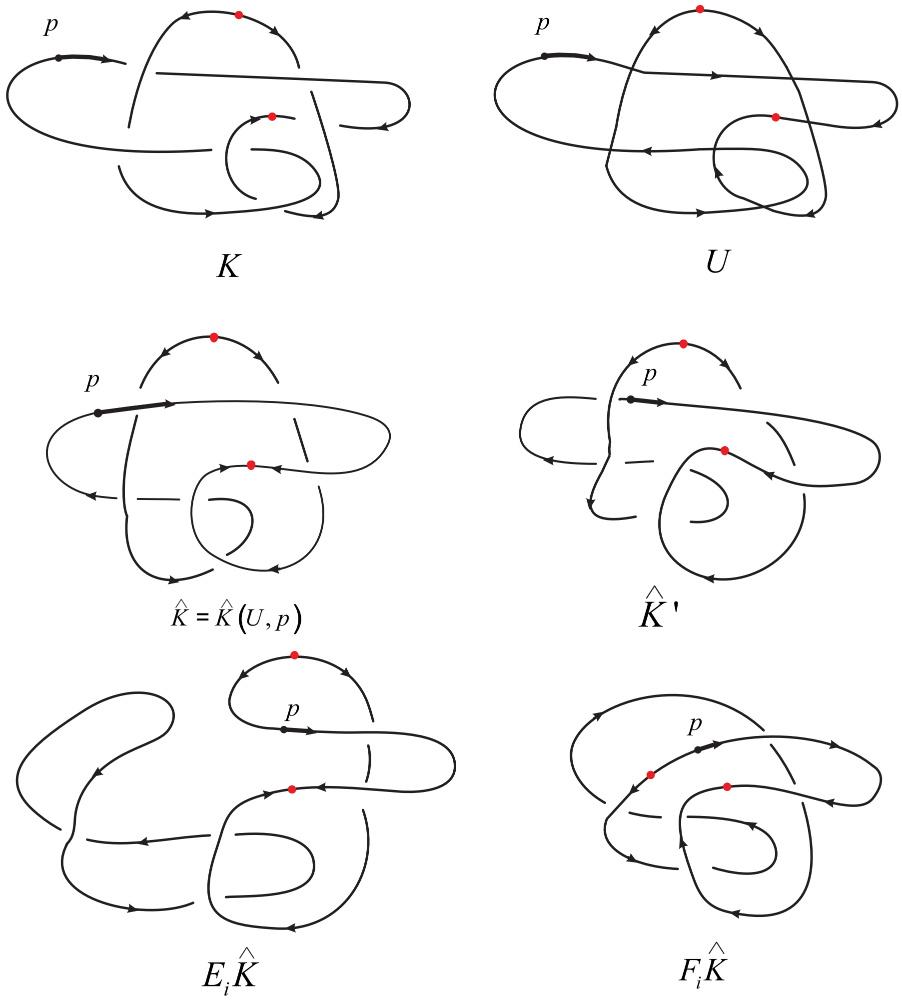

Figure 16. Normal unknot.

Remark 4.8 *1. It is obvious from Lemma 4.7 that the formula of M_K given in Definition 4.3(3) is independent of the choice of base-point.*

2. It can be proved similarly to the Lemma 6.9 in [9] that the formula given in Definition 4.3(4) is independent of the choice of base-point.

Lemma 4.9 *Assume that i is any crossing of a disoriented link diagram K. Then M_K meets the axioms:*

a. $M_K + M_{T_iK} = x(M_{E_iK} + M_{F_iK})$

b.

$$M_{I_+} = aM_{I_0}, \qquad M_{I_-} = a^{-1}M_{I_0},$$
$$M_{I'_+} = a^{-1}M_{I'_0}, \qquad M_{I'_-} = aM_{I'_0}.$$

Proof The proof of part (a) is similar to the proof of Lemma 6.10 in [9]. To confirm part (b), note that in

Definition 4.3 curls will finally be part of a disoriented knot-evaluation and these curls will not change in the corresponding normal unknot by choosing the location of the base-point. Therefore, all terms on the second side of Definition 4.3(4) contain identical copies of these curls. Then, part (b) is followed by induction. □

Lemma 4.10 *Assume that K is any disoriented link diagram and K' is another link diagram that is regularly isotopic to K. Then, $M_K = M_{K'}$. Namely, M_K is a regular isotopy invariant.*

Proof Let K be a disoriented knot. Then, the invariance under the disoriented Reidemeister moves of type II and III in Figures 17 and 18 can be demonstrated inductively by choosing the appropriate base-point.

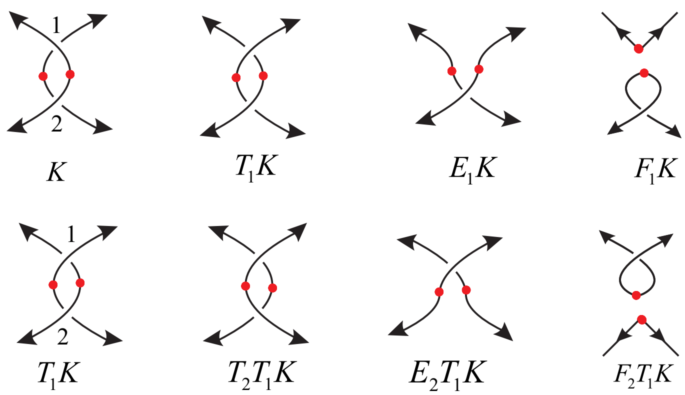

Figure 17. Switchings and eliminations of Reidemeister move of type II.

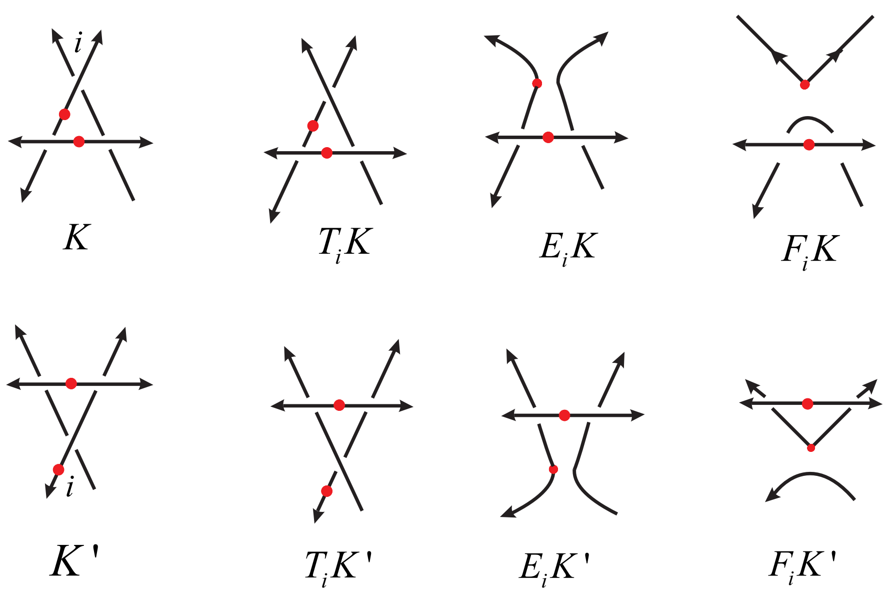

Figure 18. Switchings and eliminations of Reidemeister move of type III.

□

In case of type II, it is enough only to show that the polynomials M of the diagrams in Figure 19 are equal (Similar considerations are easily made for other moves of type II).

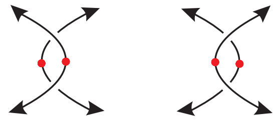

Figure 19. A Reidemeister move of type II.

On first diagram, we select the base-point as in Figure 20:

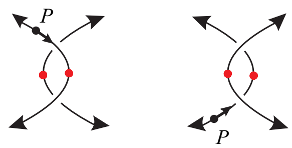

Figure 20. A Reidemeister move of type II with a base point.

Thus, the two crossings inclusive in the move are not switched by switching sequence for K. M_K is invariant under the moves of type II, because we inductively suppose that (4.4 (3)) every term in the expansion of Definition 4.3(3) is invariant under the simplifying moves of type II.

Here we supposed that invariance under simplification moves of type II is validated for every disoriented knots and links with fewer crossings than the number of crossings of K.

If the number of components of K is more than one, then we have to consider cases of the moves of type II where one of the strings is inclusive in the lifting sequence. The most likely case corresponds to choosing a base point of the form of Figure 21,

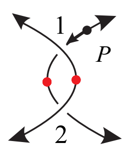

Figure 21. A Reidemeister move of type II with a base point

where the base-point must be on the underpass in order to perform the lifting sequence. However, from Figure 17 combined with Lemma 4.9 it can be seen that

$$M_K + M_{T_1K} = x(M_{E_1K} + M_{F_1K}),$$

$$M_{T_1K} + M_{T_2T_1K} = x(M_{E_2T_1K} + M_{F_2T_1K})$$

or

$$M_K - M_{T_2T_1K} = x(M_{E_1K} + M_{F_1K} - M_{E_2T_1K} - M_{F_2T_1K})$$

since $M_{E_1K} = M_{E_2T_1K}$, $M_{F_1K} = M_{F_2T_1K}$, we obtain

$$M_K = M_{T_2T_1K}.$$

In the disoriented link T_2T_1K, the move of type II will not be inclusive in a lifting sequence. Thus, invariance follows from induction as before.

Now we consider the moves of type III. In case of type III, it is sufficient only to show that the polynomials of diagrams K and K' in Figure 18 of a component of a disoriented link are equal (Similar considerations are easily made for other moves of type III). For every component of link, the ideas are the same. By choosing the base-point on K (and K'), we regulate that two of three crossings are not included in the switching sequence. Since invariant is provided under the moves of type II, we have $M_{F_iK} = M_{F_iK'}$ from Figure 18. Hence, it can be easily seen that $M_K = M_{K'}$ (and $M_{T_iK} = M_{T_iK'}$) and the invariance is achieved under the moves of type III by induction. Thus, the proof is complete.

References

[1] Altıntaş İ. Introduction to disoriented knot theory. Open Mathematics 2018; 16 (1): 346-357. https://doi.org/10.1515/math-2018-0032

[2] Altıntaş İ, Parlatıcı H. Redefining disoriented knots and diagrammatic methods. Mathematical Methods in the Applied Sciences 2022; 1-9. https://doi.org/10.1002/mma.8091

[3] Clark D, Morrison S, Walker K. Fixing the functoriality of Khovanov homology. Geometry & Topology 2009; 13 (3): 1499-1582. https://doi.org/10.2140/gt.2009.13.1499

[4] Dye HA, Kauffman LH. Virtual crossing number and the arrow polynomial. Journal of Knot Theory and its Ramifications 2009; 18 (10): 1335-1357. https://doi.org/10.1142/s0218216509007166

[5] Freyd P, Yetter D, Hoste J, Lickorish WBR, Millett K et al. A new polynomial invariant of knots and links. Bulletin of American Mathematical Society 1985; 12 (2): 239-246. https://doi.org/10.1090/s0273-0979-1985-15361-3

[6] Jones VFR. A new knot polynomial and von Neumann algebra. Notices of American Mathematical Society 1986; 33 (2): 219-225.

[7] Jones VFR. Hecke algebra representations of braid groups and link polynomials. Annals of Mathematics 1987; 126 (2): 335-388. https://doi.org/10.2307/1971403

[8] Kauffman LH. An extended bracket polynomial for virtual knots and links. Journal of Knot Theory and its Ramifications 2009; 18 (10): 1369-1422. https://doi.org/10.1142/s0218216509007543

[9] Kauffman LH. An invariant of regular isotopy. Transactions of the American Mathematical Society 1990; 318 (2): 417-471. https://doi.org/10.1090/s0002-9947-1990-0958895-7

[10] Kauffman LH. Introduction to virtual knot theory. Journal of Knot Theory and its Ramifications 2012; 21 (13): 37 pp. https://doi.org/10.1142/s021821651240007x

[11] Kauffman LH. Knots and Physics. Singapore: World Scientific, 2001.

[12] Kauffman LH. New invariants in the theory of knots. American Mathematical Monthly 1988; 95 (3): 195-242. https://doi.org/10.1080/00029890.1988.11971990

[13] Kauffman LH. State models and the Jones polynomial. Topology 1987; 26 (3): 395-407. https://doi.org/10.1016/0040-9383(87)90009-7

[14] Miyazawa Y. Magnetic graphs and an invariant for virtual links. Journal of Knot Theory and its Ramifications 2006; 15: 1319-1334. https://doi.org/10.1142/s0218216506005135

[15] Miyazawa Y. Link polynomials derived from magnetic graphs. Topology and its Applications 2010; 157: 228-246. https://doi.org/10.1016/j.topol.2009.04.062

[16] Polyak M. Minimal generating sets of Reidemeister moves. Quantum Topology 2010; 1 (4): 399-411. https://doi.org/10.4171/qt/10