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BERTRAND PARTNER P-TRAJECTORIES IN THE EUCLIDEAN 3-SPACE E^3

Zehra İŞBİLİR[1], Kahraman Esen ÖZEN[2] and Murat TOSUN[3]

[1]Department of Mathematics, Düzce University, Düzce, TÜRKİYE
[2]Department of Mathematics, Çankırı Karatekin University, Çankırı, TÜRKİYE
[3]Department of Mathematics, Sakarya University, Sakarya, TÜRKİYE

Abstract. The concept of a pair of curves, called as Bertrand partner curves, was introduced by Bertrand in 1850. Bertrand partner curves have been studied widely in the literature from past to present. In this study, we take into account of the concept of Bertrand partner trajectories according to Positional Adapted Frame (PAF) for the particles moving in 3-dimensional Euclidean space. Some characterizations are given for these trajectories with the aid of the PAF elements. Then, we obtain some special cases of these trajectories. Moreover, we provide a numerical example.

1. Introduction

The theory of curves is one of the extensive fields of study for especially differential geometry, and in the existing literature, a great number of studies have been done because of the fact that this topic is attached to the attention of a great deal of researchers. This theory investigates the geometric property of the plane and space curves by means of calculus methods. The moving frames can be seen most important structures in analyzing the calculus of curves.

Until today, many authors have been used the moving frames to investigate many special curves. For example spherical curves, Mannheim curve couple, Bertrand curve couple, involute-evolute curve couple are discussed by using the moving frames. One of these moving frames called as Positional Adapted Frame (PAF) was introduced by Özen and Tosun in 2021. The authors defined this moving

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[1] zehraisbilir@duzce.edu.tr-Corresponding author; 0000-0001-5414-5887
[2] kahramanesenozen@karatekin.edu.tr; 0000-0002-3299-6709
[3] tosun@sakarya.edu.tr; 0000-0002-4888-1412.

frame for the trajectories having non-vanishing angular momentum in Euclidean 3-space [15]. There can be found some other studies [11, 16, 19] which are performed by considering this frame.

Bertrand curve couple is one of the most popular special type curve couples. The principal normal line of one of these partner curves coincides with the principal normal line of the other partner curve at the corresponding points of these curves. This definition was given by French mathematician Joseph Louis François Bertrand in 1850 [1]. In this study, Bertrand also characterized this curve with respect to its curvature and torsion. By following the steps similar to those of Bertrand, this topic was expanded to different moving frames. For example, the studies [21], [14] and [7] expanded this topic to the type-2 Bishop frame, Darboux frame and q-frame, respectively. Also, many mathematicians presented various studies about the concept of Bertrand curve couple with different perspectives. Some of them can be found in [3, 10, 12, 17, 20]. In this study, we will consider this topic with respect to the Positional Adapted Frame.

This study is organized as follows. In Section 2, we review some required information to understand the ensuing section. In Section 3, we deal with Bertrand partner trajectories according to Positional Adapted Frame in 3-dimensional Euclidean space. We call these trajectories as Bertrand partner P-trajectories. We examine the relationships between the PAF elements of the aforesaid partners. Also, we give the relations between the Serret-Frenet basis vectors of Bertrand partner P-trajectories. Moreover, we get the necessary conditions in terms of the PAF curvatures of other to be an osculating curve for one of these partners. Lastly, we provide a numerical example so that the readers can visualize the Bertrand partner P-trajectories.

2. Basic Concepts

In this section, we have reviewed some required and fundamental concepts to disambiguate the ensuing section of the paper.

In Euclidean 3-space E^3, let $\boldsymbol{U} = (u_1, u_2, u_3), \boldsymbol{V} = (v_1, v_2, v_3) \in \mathbb{R}^3$ be given. The standard dot product of these vectors and the norm of $\boldsymbol{U}$ are given as $\langle \boldsymbol{U}, \boldsymbol{V} \rangle = u_1v_1 + u_2v_2 + u_3v_3$ and $\|\boldsymbol{U}\| = \sqrt{\langle \boldsymbol{U}, \boldsymbol{U} \rangle}$, respectively. A differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ is called as a unit speed curve if $\left\| \frac{d\alpha}{ds} \right\| = 1$ holds for each $s \in I$. In that case, s is called as arc-length parameter of the curve α. If the derivative of a differentiable curve never vanishes along this curve, it is said to be a regular curve. Any regular curve always has a parameterization such that it will be a unit speed curve [18]. Note that the symbol prime "$\prime$" will be used to indicate the differentiation according to the arc-length parameter s in the rest of the paper.

Let us take into consideration a point particle P of a constant mass moves on a unit speed regular curve $\alpha = \alpha(s)$. The base vectors of the Serret-Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ of α are defined by the equations $\mathbf{T}(s) = \alpha'(s)$,

$\mathbf{N}(s)=\frac{\alpha''(s)}{\|\alpha''(s)\|}$, $\mathbf{B}(s)=\mathbf{T}(s)\wedge\mathbf{N}(s)$. The base vectors $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are called as unit tangent vector, principal normal vector and binormal vector, respectively. The Serret-Frenet derivative formulas are expressed as in the following:

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{N}'(s)\\ \mathbf{B}'(s)\end{pmatrix}=\begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix} \tag{1}$$

where $\kappa(s)=\|\mathbf{T}'(s)\|$ is the curvature and $\tau(s)=-\langle\mathbf{B}'(s),\mathbf{N}(s)\rangle$ is the torsion [18]. We must emphasize that the curvature κ never vanishes for the curves we will consider in this paper.

On the other hand, it is well known that the vector product of the position vector $\mathbf{x}=\langle\alpha(s),\mathbf{T}(s)\rangle\mathbf{T}(s)+\langle\alpha(s),\mathbf{N}(s)\rangle\mathbf{N}(s)+\langle\alpha(s),\mathbf{B}(s)\rangle\mathbf{B}(s)$ and the linear momentum vector $\mathbf{p}(t)=m\left(\frac{ds}{dt}\right)\mathbf{T}(s)$ of the particle P yields the angular momentum vector of P about the origin as $\mathbf{H}^O=m\langle\alpha(s),\mathbf{B}(s)\rangle\left(\frac{ds}{dt}\right)\mathbf{N}(s)-m\langle\alpha(s),\mathbf{N}(s)\rangle\left(\frac{ds}{dt}\right)\mathbf{B}(s)$. Here m and t denote the constant mass of P and the time [4,9]. Let this vector not equal to zero vector during the motion of P. Making this supposition assures that the coefficient functions $\langle\alpha(s),\mathbf{N}(s)\rangle$ and $\langle\alpha(s),\mathbf{B}(s)\rangle$ of the position vector $\mathbf{x}$ do not equal to zero at the same time. Then, one can easily say that the tangent line of $\alpha=\alpha(s)$ does not pass through the origin along the trajectory of P. Take into account of the vector whose initial point is the foot of the perpendicular (from origin to instantaneous rectifying plane $Sp\{\mathbf{T}(s),\mathbf{B}(s)\}$) and endpoint is the foot of the perpendicular (from origin to instantaneous osculating plane $Sp\{\mathbf{T}(s),\mathbf{N}(s)\}$). The unit vector in direction of the equivalent of the aforementioned vector at the point $\alpha(s)$ determines the PAF basis vector $\mathbf{Y}(s)$. The other PAF basis vector $\mathbf{M}(s)$ is obtained by the vector product $\mathbf{Y}(s)\wedge\mathbf{T}(s)$. Consequently, the vectors

$$\begin{aligned}
\mathbf{T}(s)&=\mathbf{T}(s),\\
\mathbf{M}(s)&=\frac{\langle\alpha(s),\mathbf{B}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{N}(s)+\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{B}(s),\\
\mathbf{Y}(s)&=\frac{\langle-\alpha(s),\mathbf{N}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{N}(s)+\frac{\langle\alpha(s),\mathbf{B}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{B}(s),
\end{aligned}$$

form the Positional Adapted Frame $\{\mathbf{T}(s),\mathbf{M}(s),\mathbf{Y}(s)\}$ (see [15] for more details on PAF).

The relation between the Serret-Frenet frame and PAF is as in the following:

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\Omega(s) & -\sin\Omega(s)\\ 0 & \sin\Omega(s) & \cos\Omega(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix} \tag{2}$$

where $\Omega(s)$ is the angle between the vector $\mathbf{B}(s)$ and the vector $\mathbf{Y}(s)$ which is positively oriented from the vector $\mathbf{B}(s)$ to vector $\mathbf{Y}(s)$. On the other hand, the derivative formulas of PAF are presented as follows [15]:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{M}'(s) \\ \mathbf{Y}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{M}(s) \\ \mathbf{Y}(s) \end{pmatrix} \tag{3}$$

where

$$k_1(s) = \kappa(s)\cos\Omega(s),$$
$$k_2(s) = \kappa(s)\sin\Omega(s),$$
$$k_3(s) = \tau(s) - \Omega'(s).$$

Here, the rotation angle $\Omega(s)$ is determined by means of the following equation:

$$\Omega(s) = \begin{cases} \arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right) \;\; if \;\; \langle\alpha(s),\mathbf{B}(s)\rangle > 0, \\ \arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right) + \pi \;\; if \;\; \langle\alpha(s),\mathbf{B}(s)\rangle < 0, \\ -\frac{\pi}{2} \;\; if \;\; \langle\alpha(s),\mathbf{B}(s)\rangle = 0\;, \;\; \langle\alpha(s),\mathbf{N}(s)\rangle > 0, \\ \frac{\pi}{2} \;\; if \;\; \langle\alpha(s),\mathbf{B}(s)\rangle = 0\;, \;\; \langle\alpha(s),\mathbf{N}(s)\rangle < 0. \end{cases}$$

The elements of the set $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s), k_1(s), k_2(s), k_3(s)\}$ are called as PAF apparatuses of $\alpha = \alpha(s)$ [15].

Note that PAF is a generic adapted moving frame just like Bishop frame [2], Darboux frame [6], B-Darboux frame [8] etc. Generic adapted moving frames are obtained from Serret-Frenet frame by a rotation (see [5] for more details on generic adapted moving frame). Since the analytical approach is used to determine the rotation angle in PAF, the rotation angle can be easily determined, while in many other moving frames, the determination of the angle is based on integral calculations. These calculations often cause difficulties for researchers. Also, PAF enables the researchers to study the kinematics of a moving particle and the differential geometry of this particle at the same time. Moreover, PAF contains information about the position vector of the moving particle. When viewed from this aspect, it is a useful tool for the researchers studying on kinematics and inverse kinematics.

Now we give the definition of the osculating curve in 3-dimensional Euclidean space since we will discuss this topic in the next section. A curve $\beta = \beta(s)$ is called as osculating curve if its position vector always lies in its osculating plane. One can find more details on this topic in [13].

Theorem 1. *[15] Let $\alpha = \alpha(s)$ be the unit speed parameterization of the trajectory. Then, α is an osculating curve if and only if $k_1 = 0$.*

More details can be found in the studies [11, 15, 16, 19] for Positional Adapted Frame (PAF).

3. Bertrand Partner P-Trajectories

In this section, we introduce the Bertrand partner P-trajectories and give some characterizations of them. Furthermore, we provide an example in order to illustrate this topic.

Definition 1. *Let Q and $\breve{Q}$ be the moving point particles of constant masses in the Euclidean 3-space. Show the unit speed parameterization of the trajectories of Q and $\breve{Q}$ with $\alpha = \alpha(s)$ and $\breve{\alpha} = \breve{\alpha}(\breve{s})$, respectively. Let the PAF apparatus of the trajectories α and $\breve{\alpha}$ be represented by $\{\mathbf{T}, \mathbf{M}, \mathbf{Y}, k_1, k_2, k_3\}$ and $\left\{\breve{\mathbf{T}}, \breve{\mathbf{M}}, \breve{\mathbf{Y}}, \breve{k}_1, \breve{k}_2, \breve{k}_3\right\}$, respectively. If the PAF base vector $\mathbf{M}$ coincides with the PAF base vector $\breve{\mathbf{M}}$ at the corresponding points of the trajectories α and $\breve{\alpha}$, in this case α is said to be a Bertrand partner P-trajectory of $\breve{\alpha}$. Moreover, the pair $\{\alpha, \breve{\alpha}\}$ is called as a Bertrand P-pair.*

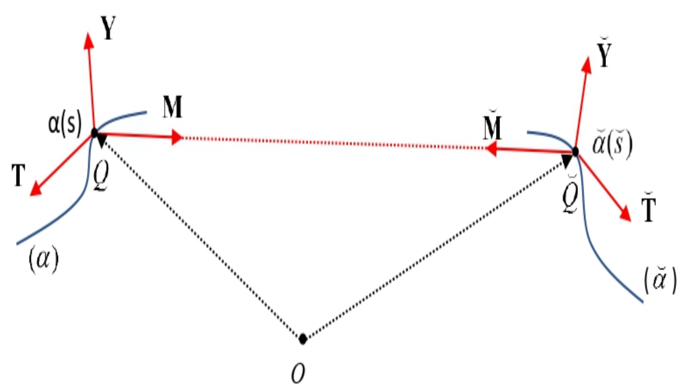

Figure 1. Bertrand partner P-trajectories

According to the definition of Bertrand P-pair, we get the following matrix equation

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{M} \\ \mathbf{Y} \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \breve{\mathbf{T}} \\ \breve{\mathbf{M}} \\ \breve{\mathbf{Y}} \end{pmatrix} \quad (4)$$

where ϕ is the angle between the tangent vectors $\mathbf{T}$ and $\breve{\mathbf{T}}$.

Theorem 2. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be any Bertrand P-pair in E^3. In that case, the distance between the corresponding points of α and $\breve{\alpha}$ is constant.*

Proof. By the definition of Bertrand P-trajectories, we can write:

$$\alpha(s) = \breve{\alpha}(\breve{s}) + \psi(\breve{s})\breve{\mathbf{M}}(\breve{s}) \tag{5}$$

where ψ is a real valued smooth function of $\breve{s}$ (see Figure 1). By taking the derivative of the equation (5) with respect to $\breve{s}$ and considering the PAF derivative formulas (3), we get:

$$\mathbf{T}\frac{ds}{d\breve{s}} = \left(1 - \psi\breve{k}_1\right)\breve{\mathbf{T}} + \psi'\breve{\mathbf{M}} + \psi\breve{k}_3\breve{\mathbf{Y}}. \tag{6}$$

Since $\mathbf{T}$, $\breve{\mathbf{T}}$ and $\breve{\mathbf{Y}}$ are orthogonal to $\breve{\mathbf{M}}$, and also $\breve{\mathbf{M}}$ is a unit vector, we have $\psi' = 0$ with the help of the inner product. Therefore, ψ is a non-zero constant and the equation (6) becomes:

$$\mathbf{T}\frac{ds}{d\breve{s}} = \left(1 - \psi\breve{k}_1\right)\breve{\mathbf{T}} + \psi\breve{k}_3\breve{\mathbf{Y}}. \tag{7}$$

In the light of these results, the distance between the corresponding points of the trajectories can be given as:

$$d(\alpha(s), \breve{\alpha}(\breve{s})) = \|\alpha(s) - \breve{\alpha}(\breve{s})\| = \left\|\psi\breve{\mathbf{M}}\right\| = \mid\psi\mid.$$

Therefore, we can say that the distance between each corresponding points of α and $\breve{\alpha}$ is constant. □

Theorem 3. *Let* $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ *be any Bertrand P-pair in* E^3. *Then, the equation*

$$\frac{d}{ds}(\cos\phi) = k_2\left\langle \mathbf{Y}, \breve{\mathbf{T}}\right\rangle + \breve{k}_2\frac{d\breve{s}}{ds}\left\langle \mathbf{T}, \breve{\mathbf{Y}}\right\rangle$$

is satisfied.

Proof. Since ϕ is the angle between the tangent vectors $\mathbf{T}$ and $\breve{\mathbf{T}}$, one can easily write $\left\langle \mathbf{T}, \breve{\mathbf{T}}\right\rangle = \|\mathbf{T}\|\left\|\breve{\mathbf{T}}\right\|\cos\phi = \cos\phi$. Let us differentiate this equation with respect to s. Thus, we get:

$$\begin{aligned}\frac{d}{ds}(\cos\phi) &= \frac{d}{ds}\left\langle \mathbf{T}, \breve{\mathbf{T}}\right\rangle \\ &= \left\langle k_1\mathbf{M} + k_2\mathbf{Y}, \breve{\mathbf{T}}\right\rangle + \left\langle \mathbf{T}, (\breve{k}_1\breve{\mathbf{M}} + \breve{k}_2\breve{\mathbf{Y}})\frac{d\breve{s}}{ds}\right\rangle.\end{aligned}$$

This equation gives us the desired result. □

Corollary 1. *The angles between the tangent vectors at the corresponding points of a Bertrand P-pair is generally not constant.*

Theorem 4. *Let $\{\alpha=\alpha(s),\ \breve{\alpha}=\breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, the following relations*

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix}=\begin{pmatrix}\left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds} & 0 & \psi\breve{k}_3\frac{d\breve{s}}{ds}\\ 0 & 1 & 0\\ -\psi\breve{k}_3\frac{d\breve{s}}{ds} & 0 & \left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds}\end{pmatrix}\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{M}}\\ \breve{\mathbf{Y}}\end{pmatrix} \tag{8}$$

are satisfied between the PAF vectors of α and $\breve{\alpha}$.

Proof. Suppose that $\{\alpha,\breve{\alpha}\}$ is a Bertrand P-pair in E^3. By using the equations (4) and (7), we get:

$$\cos\phi\frac{ds}{d\breve{s}}\breve{\mathbf{T}}-\sin\phi\frac{ds}{d\breve{s}}\breve{\mathbf{Y}}=\left(1-\psi\breve{k}_1\right)\breve{\mathbf{T}}+\psi\breve{k}_3\breve{\mathbf{Y}}.$$

The last equation gives us the following:

$$\begin{cases}\cos\phi=\left(1-\psi\breve{k}_1\right)\dfrac{d\breve{s}}{ds},\\ \sin\phi=-\psi\breve{k}_3\dfrac{d\breve{s}}{ds}.\end{cases} \tag{9}$$

If we substitute the equation (9) in the equation (4), we obtain the desired result. □

Corollary 2. *Let $\{\alpha=\alpha(s),\breve{\alpha}=\breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, we have:*

$$\tan\phi=\frac{-\psi\breve{k}_3}{1-\psi\breve{k}_1} \tag{10}$$

where ϕ is the angle between $\mathbf{T}$ and $\breve{\mathbf{T}}$.

Corollary 3. *Let $\{\alpha=\alpha(s),\breve{\alpha}=\breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then,*

$$\int\cos\phi ds+\psi\int\breve{k}_1 d\breve{s}=\breve{s}+c_1$$

where c_1 denotes the integration constant.

Corollary 4. *Let $\{\alpha=\alpha(s),\breve{\alpha}=\breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, the equality*

$$\int\sin\phi ds+\psi\int\breve{k}_3 d\breve{s}=0$$

holds.

Theorem 5. *Let $\{\alpha=\alpha(s),\breve{\alpha}=\breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3 and their Serret-Frenet apparatuses be $\{\mathbf{T},\mathbf{N},\mathbf{B},\kappa,\tau\}$ and $\left\{\breve{\mathbf{T}},\breve{\mathbf{N}},\breve{\mathbf{B}},\breve{\kappa},\breve{\tau}\right\}$, respectively.*

Then, the relations between the Serret-Frenet vectors of this pair are given as:

$$\begin{aligned}
\breve{\mathbf{T}} &= \left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds}\mathbf{T}-\psi\breve{k}_3\sin\Omega\frac{d\breve{s}}{ds}\mathbf{N}-\psi\breve{k}_3\cos\Omega\frac{d\breve{s}}{ds}\mathbf{B},\\
\breve{\mathbf{N}} &=\psi\breve{k}_3\sin\breve{\Omega}\frac{d\breve{s}}{ds}\mathbf{T}+\left(\cos\breve{\Omega}\cos\Omega+\left(1-\psi\breve{k}_1\right)\sin\breve{\Omega}\sin\Omega\frac{d\breve{s}}{ds}\right)\mathbf{N}\\
&\quad+\left(-\cos\breve{\Omega}\sin\Omega+\left(1-\psi\breve{k}_1\right)\sin\breve{\Omega}\cos\Omega\frac{d\breve{s}}{ds}\right)\mathbf{B},\\
\breve{\mathbf{B}} &=\psi\breve{k}_3\cos\breve{\Omega}\frac{d\breve{s}}{ds}\mathbf{T}+\left(-\sin\breve{\Omega}\cos\Omega+\left(1-\psi\breve{k}_1\right)\cos\breve{\Omega}\sin\Omega\frac{d\breve{s}}{ds}\right)\mathbf{N}\\
&\quad+\left(\sin\breve{\Omega}\sin\Omega+\left(1-\psi\breve{k}_1\right)\cos\breve{\Omega}\cos\Omega\frac{d\breve{s}}{ds}\right)\mathbf{B},
\end{aligned}$$

where Ω is the angle between the vectors $\mathbf{B}$ and $\mathbf{Y}$ and also, $\breve{\Omega}$ is the angle between the vectors $\breve{\mathbf{B}}$ and $\breve{\mathbf{Y}}$.

Proof. Using the equation (2), we can write:

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix}=\begin{pmatrix}1&0&0\\ 0&\cos\Omega&-\sin\Omega\\ 0&\sin\Omega&\cos\Omega\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}\tag{11}$$

and also

$$\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{N}}\\ \breve{\mathbf{B}}\end{pmatrix}=\begin{pmatrix}1&0&0\\ 0&\cos\breve{\Omega}&\sin\breve{\Omega}\\ 0&-\sin\breve{\Omega}&\cos\breve{\Omega}\end{pmatrix}\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{M}}\\ \breve{\mathbf{Y}}\end{pmatrix}.\tag{12}$$

On the other hand, by using the equation (8), we get:

$$\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{M}}\\ \breve{\mathbf{Y}}\end{pmatrix}=\begin{pmatrix}\left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds}&0&-\psi\breve{k}_3\frac{d\breve{s}}{ds}\\ 0&1&0\\ \psi\breve{k}_3\frac{d\breve{s}}{ds}&0&\left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix}.\tag{13}$$

If the equation (13) is substituted into the equation (12), then

$$\begin{pmatrix}\breve{\mathbf{T}}\\ \breve{\mathbf{N}}\\ \breve{\mathbf{B}}\end{pmatrix}=\begin{pmatrix}\left(1-\psi\breve{k}_1\right)\frac{d\breve{s}}{ds}&0&-\psi\breve{k}_3\frac{d\breve{s}}{ds}\\ \psi\breve{k}_3\sin\breve{\Omega}\frac{d\breve{s}}{ds}&\cos\breve{\Omega}&\left(1-\psi\breve{k}_1\right)\sin\breve{\Omega}\frac{d\breve{s}}{ds}\\ \psi\breve{k}_3\cos\breve{\Omega}\frac{d\breve{s}}{ds}&-\sin\breve{\Omega}&\left(1-\psi\breve{k}_1\right)\cos\breve{\Omega}\frac{d\breve{s}}{ds}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{M}\\ \mathbf{Y}\end{pmatrix}\tag{14}$$

is found. By using the equation (11) in the equation (14), we complete the proof. □

Theorem 6. *Let $\{\alpha=\alpha(s),\breve{\alpha}=\breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, the following relations*

(1) $k_1 = \dfrac{\breve{k}_1 - \psi\breve{k}_1^{\,2} - \psi\breve{k}_3^{\,2}}{1 - 2\psi\breve{k}_1 + \psi^2\left(\breve{k}_1^{\,2} + \breve{k}_3^{\,2}\right)}$,

(2) $\breve{k}_1 = \dfrac{k_1 - \eta k_1^2 - \eta k_3^2}{1 - 2\eta k_1 + \eta^2\left(k_1^2 + k_3^2\right)}$,

are satisfied between k_1, k_3, $\breve{k}_1$ *and* $\breve{k}_3$*. Here* η *is a constant satisfying* $\mid \eta \mid = \mid \psi \mid$*.*

Proof. (1) Assume that $\{\alpha, \breve{\alpha}\}$ is a Bertrand P-pair in E^3. Via the equation (9) and the equality $\cos^2\phi + \sin^2\phi = 1$, we get:

$$\left(\frac{d\breve{s}}{ds}\right)^2\left(\left(1 - \psi\breve{k}_1\right)^2 + \psi^2\breve{k}_3^{\,2}\right) = 1.$$

Hence, we have:

$$\left(\frac{ds}{d\breve{s}}\right)^2 = 1 - 2\psi\breve{k}_1 + \psi^2\left(\breve{k}_1^{\,2} + \breve{k}_3^{\,2}\right). \tag{15}$$

On the other hand, if we differentiate the equation (7) with respect to $\breve{s}$ and use the PAF derivative formulas, we obtain:

$$\begin{aligned}\frac{d^2s}{d\breve{s}^2}\mathbf{T} + k_1\left(\frac{ds}{d\breve{s}}\right)^2\mathbf{M} + k_2\left(\frac{ds}{d\breve{s}}\right)^2\mathbf{Y} = &\left(-\psi\breve{k}_1{}' - \psi\breve{k}_2\breve{k}_3\right)\breve{\mathbf{T}}\\ &+ \left(\breve{k}_1\left(1 - \psi\breve{k}_1\right) - \psi\breve{k}_3^{\,2}\right)\breve{\mathbf{M}}\\ &+ \left(\breve{k}_2\left(1 - \psi\breve{k}_1\right) + \psi\breve{k}_3{}'\right)\breve{\mathbf{Y}}.\end{aligned} \tag{16}$$

By taking into consideration the equation (16) and utilizing the definition of Bertrand P-pair, we get:

$$k_1\left(\frac{ds}{d\breve{s}}\right)^2 = \left(1 - \psi\breve{k}_1\right)\breve{k}_1 - \psi\breve{k}_3^{\,2}. \tag{17}$$

From the equations (15) and (17), one can easily see the desired result.

(2) According to the definition of the Bertrand P-pair, we can write:

$$\breve{\alpha}(\breve{s}) = \alpha(s) + \eta\mathbf{M}(s)$$

where η is a constant satisfying $\mid \eta \mid = \mid \psi \mid$ (see Figure 1). Let us take the derivative of this equation with respect to s twice. In that case, we obtain:

$$\breve{\mathbf{T}}\frac{d\breve{s}}{ds} = (1 - \eta k_1)\mathbf{T} + \eta k_3\mathbf{Y} \tag{18}$$

and

$$\begin{aligned}\frac{d^2\breve{s}}{ds^2}\breve{\mathbf{T}} + \breve{k}_1\left(\frac{d\breve{s}}{ds}\right)^2\breve{\mathbf{M}} + \breve{k}_2\left(\frac{d\breve{s}}{ds}\right)^2\breve{\mathbf{Y}} = &\left(-\eta k_1' - \eta k_2k_3\right)\mathbf{T}\\ &+ \left(k_1(1 - \eta k_1) - \eta k_3^2\right)\mathbf{M}\\ &+ \left(k_2(1 - \eta k_1) + \eta k_3'\right)\mathbf{Y}.\end{aligned} \tag{19}$$

On the other hand, we can write $\breve{\mathbf{T}} = \cos\phi\, \mathbf{T} + \sin\phi\, \mathbf{Y}$ by the equation (4). Then, by using the equation (18), we find:

$$\cos\phi \frac{d\breve{s}}{ds}\mathbf{T} + \sin\phi \frac{d\breve{s}}{ds}\mathbf{Y} = (1 - \eta k_1)\, \mathbf{T} + \eta k_3 \mathbf{Y}.$$

Hence, we get the equations $\cos\phi \frac{d\breve{s}}{ds} = 1 - \eta k_1$ and $\sin\phi \frac{d\breve{s}}{ds} = \eta k_3$. These equations give us the equation:

$$\left(\frac{d\breve{s}}{ds}\right)^2 = 1 - 2\eta k_1 + \eta^2 \left(k_1^2 + k_3^2\right). \tag{20}$$

Moreover, by taking the inner product of the vectors at the right and left sides of the equation (19) with the vector $\mathbf{M}$, we have:

$$\breve{k}_1 \left(\frac{d\breve{s}}{ds}\right)^2 = k_1 - \eta k_1^2 - \eta k_3^2. \tag{21}$$

Therefore, we obtain the desired result by using the equation (20).

□

Thanks to the Theorem 1 and Theorem 6, we can attain the following corollaries.

Corollary 5. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in Euclidean 3-space E^3. If $\breve{k}_1 = \breve{k}_3 = 0$, then $k_1 = 0$.*

Corollary 6. *Let $\{\alpha = \alpha(s), \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in Euclidean 3-space E^3. If $k_1 = k_3 = 0$, then $\breve{k}_1 = 0$.*

Corollary 7. *Let $\{\alpha = \alpha(s),\ \breve{\alpha} = \breve{\alpha}(\breve{s})\}$ be a Bertrand P-pair in E^3. Then, the following mathematical expressions hold:*

(1) *α is an osculating curve if and only if* $\frac{\breve{k}_1 - \psi \breve{k}_1^{\,2} - \psi \breve{k}_3^{\,2}}{1 - 2\psi\breve{k}_1 + \psi^2\left(\breve{k}_1^{\,2} + \breve{k}_3^{\,2}\right)} = 0,$

(2) *$\breve{\alpha}$ is an osculating curve if and only if* $\frac{k_1 - \eta k_1^2 - \eta k_3^2}{1 - 2\eta k_1 + \eta^2\left(k_1^2 + k_3^2\right)} = 0.$

Example 1. *In the Euclidean 3-space, suppose that a point particle Q moves on the trajectory*

$$\begin{aligned} \alpha &: (0, \pi/2) \to E^3 \\ s &\mapsto \alpha(s) = \left(\frac{8}{17}\cos 2s, \frac{12}{17} - \sin 2s, -\frac{15}{17}\cos 2s\right). \end{aligned} \tag{22}$$

By straightforward calculations, we get the following Serret-Frenet apparatus:

$$\begin{cases}\mathbf{T}(s)=\left(-\frac{8}{17}\sin 2s,-\cos s,\frac{15}{17}\sin 2s\right)\\ \mathbf{N}(s)=\left(-\frac{8}{17}\cos 2s,\sin 2s,\frac{15}{17}\cos 2s\right)\\ \mathbf{B}(s)=\left(-\frac{15}{17},0,-\frac{8}{17}\right)\end{cases} \quad \textit{and} \quad \begin{cases}\kappa(s)=1\\ \tau(s)=0.\end{cases}$$

Since $\langle\alpha(s),\mathbf{B}(s)\rangle=0$ *and* $\langle\alpha(s),\mathbf{N}(s)\rangle=-1+\frac{12}{17}\sin 2s<0$, *we get* $\Omega=\frac{\pi}{2}$. *Then, the elements of PAF are found as:*

$$\begin{cases}\mathbf{T}(s)=\left(-\frac{8}{17}\sin 2s,-\cos 2s,\frac{15}{17}\sin 2s\right)\\ \mathbf{M}(s)=\left(\frac{15}{17},0,\frac{8}{17}\right)\\ \mathbf{Y}(s)=\left(-\frac{8}{17}\cos 2s,\sin 2s,\frac{15}{17}\cos 2s\right)\end{cases} \quad \textit{and} \quad \begin{cases}k_1(s)=0\\ k_2(s)=1\\ k_3(s)=0.\end{cases}$$

Therefore, Bertrand partner P-trajectory of α *can be given as:*

$$\breve{\alpha}(s)=\left(\frac{8}{17}\cos 2s+\eta\frac{15}{17},\frac{12}{17}-\sin 2s,-\frac{15}{17}\cos 2s+\eta\frac{8}{17}\right) \tag{23}$$

by means of the equality $\breve{\alpha}(s)=\alpha(s)+\eta\mathbf{M}(s)$.

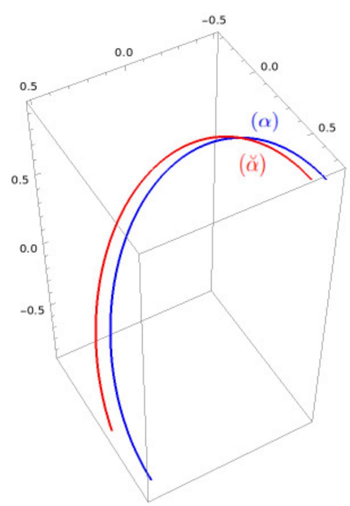

FIGURE 2. The trajectories α and $\breve{\alpha}$ given in (22) and (23)

In the Figure 2, the trajectories $\alpha = \alpha(s)$ (blue) and $\breve{\alpha} = \breve{\alpha}(s)$ (red) can be seen. Here we take $\eta = 0.1$.

On the other hand, by using the Theorem 6 and Corollary 7, we get $\breve{k}_1 = 0$. So, we can conclude that the trajectory $\breve{\alpha}$ is an osculating curve. It should be noted that the Figure 2 is drawn by utilizing the website Wolfram Mathematica (Wolfram Cloud).

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References

[1] Bertrand, J., Mémoire sur la théorie des courbes à double courbure, *J. Math. Pures Appl.*, (1850) 332–350.

[2] Bishop, R. L., There is more than one way to frame a curve, *Amer. Math. Monthly*, 82(3) (1975) 246–251. https://doi.org/10.2307/2319846

[3] Burke, J. F., Bertrand curves associated with a pair of curves, *Math. Mag.*, 34(1) (1960), 60–62. https://doi.org/10.2307/2687860

[4] Casey, J., Siacci's resolution of the acceleration vector for a space curve, *Meccanica*, 46 (2011), 471–476. https://doi.org/10.1007/s11012-010-9296-x

[5] da Silva, L. C., Differential Geometry of Rotation Minimizing Frames, Spherical Curves, and Quantum Mechanics of a Constrained Particle, Ph.D Thesis, 2017, (2018), arXiv preprint, arXiv:1806.08830.

[6] Darboux, G., Leçons Sur La Théorie Générale Des Surfaces I-II-III-IV, Gauthier Villars, Paris, 1896.

[7] Dede, M., Ekici, C., Directional Bertrand curves, *Gazi Univ. J. Sci.*, 31(1) (2018), 202–211.

[8] Dede, M., Aslan, M. Ç., Ekici, C., On a variational problem due to the B-Darboux frame in Euclidean 3-space, *Math. Methods Appl. Sci.*, 44(17) (2021), 12630–12639. https://doi.org/10.1002/mma.7567

[9] Goldstein, H., Poole, C. P., Safko, J. L., Classical Mechanics, Vol. 2, Reading, MA: Addison-Wesley, 1950.

[10] Görgülü, A., A generalization of the Bertrand curves as general inclined curves in E^n, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 35(01.02) (1986), 54–60. https://doi.org/10.1501/Commua1_0000000254

[11] Gürbüz, N. E., The evolution of an electric field with respect to the type-1 PAF and the PAFORS frames in $\mathbb{R}^3_1$, *Optik*, 250(1) (2022), 168285. https://doi.org/10.1016/j.ijleo.2021.168285

[12] Izumiya, S., Takeuchi, N., Generic properties of helices and Bertrand curves, *J. Geom.*, 74 (2002), 97–109. https://doi.org/10.1007/PL00012543

[13] İlarslan, K., Nesovic, E., Some characterizations of osculating curves in the Euclidean spaces, *Demonstr. Math.*, 41(4) (2008), 931–939. https://doi.org/10.1515/dema-2008-0421

[14] Kazaz, M., Uğurlu, M. M., Önder, M., Oral, S., Bertrand partner D-curves in the Euclidean 3-space E^3, *Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi*, 16(1) (2016), 76–83. https://doi.org/10.5578/fmbd.25270

[15] Özen, K. E., Tosun, M., A new moving frame for trajectories with non-vanishing angular momentum, *J. Math. Sci. Model.*, 4(1) (2021), 7–18. https://doi.org/10.33187/jmsm.869698

[16] Özen, K. E., Tosun, M., Trajectories generated by special Smarandache curves according to Positional Adapted Frame, *Karamanoğlu Mehmetbey University Journal of Engineering and Natural Sciences*, 3(1) (2021), 15–23.

[17] Papaioannou, S. G., Kiritsis, D., An application of Bertrand curves and surfaces to CADCAM, *Computer-Aided Design*, 17(8) (1985), 348–352. https://doi.org/10.1016/0010-4485(85)90025-9

[18] Shifrin T., Differential Geometry: A First Course in Curves and Surfaces, University of Georgia, Preliminary Version, 2008.

[19] Solouma, E. M., Characterization of Smarandache trajectory curves of constant mass point particles as they move along the trajectory curve via PAF, *Bulletin of Mathematical Analysis and Applications*, 13(4) (2021), 14–30. https://doi.org/10.54671/bmaa-2021-4-2

[20] Whittemore, J. K., Bertrand curves and helices, *Duke Math. J.*, 6(1) (1940), 235–245. https://doi.org/10.1215/S0012-7094-40-00618-4

[21] Yerlikaya, F., Karaahmetoğlu, S., Aydemir, İ., On the Bertrand B-pair curve in 3-dimensional Euclidean space, *Journal of Science and Arts*, 3(36) (2016), 215–224.