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Pauli–Leonardo quaternions *

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Abstract: In this study, we define Pauli–Leonardo quaternions by taking the coefficients of the Pauli quaternions as Leonardo numbers. We give the recurrence relation, Binet formula, generating function, exponential generating function, some special equalities, and the sum properties of these novel quaternions. In addition, we investigate the interrelations between Pauli–Leonardo quaternions and the Pauli–Fibonacci, Pauli–Lucas quaternions. Moreover, we create some algorithms that determine the terms of the Pauli–Leonardo quaternions. Finally, we generate the matrix representations of the Pauli–Leonardo quaternions and $\mathbb{R}$–linear transformations.

Keywords: Leonardo numbers, Pauli quaternions, Pauli–Leonardo quaternions.
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1 Introduction

In the last ten years, extensive research has been carried out on both the Fibonacci sequence and its generalizations, as well as the quaternions with Fibonacci coefficients and the generalizations of these quaternions [2, 12, 15, 20, 21, 32, 33, 45]. Many researchers have developed a great interest in the Fibonacci numbers because of the concept of the golden ratio obtained by the ratio of consecutive terms of the Fibonacci sequence, which may occur in nature or in a variety of other fields such as architecture, finance, art, music, etc.

As it is well known, the Fibonacci numbers are a sequence of integers in which every number is the sum of two numbers preceding it in the sequence. Thus, the n-th Fibonacci number denoted by F_n satisfies the recurrence relation: $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. On the other hand, the n-th Lucas number denoted by L_n holds the recurrence relation: $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$ with the initial conditions $L_0 = 2$ and $L_1 = 1$ [13, 26, 27, 39]. While the recurrence relations of Fibonacci and Lucas numbers are the same, some new numbers with different recurrence relations and different initial conditions have been introduced. Starting from a similar point of view the features analogical to those obtained from Fibonacci numbers have been examined for these numbers. One of these number sequences is the Leonardo numbers sequence, which has been first studied by Catarino and Borges [7] (A001595 in [39]). The n-th Leonardo number denoted by Le_n satisfies the following recurrence relations:

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \text{ for all } n \geq 2 \tag{1}$$

or

$$Le_{n+1} = 2Le_n - Le_{n-2}, \text{ for all } n \geq 2 \tag{2}$$

with the initial conditions $Le_0 = 1, Le_1 = 1$ [7].

In the following Table 1, some values can be seen concerning these three types of special recurrence sequences [3, 7, 26, 39].

Table 1. Some values for Fibonacci, Lucas and Leonardo numbers

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	...
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	...
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843	...
Le_n	1	1	3	5	9	15	25	41	67	109	177	287	465	753	1219	...

The studies on the Leonardo numbers, generalizations and matrices of these numbers have been gathered speed in recent years. For instance, Alp and Koçer have examined some properties of the Leonardo numbers in [3]. Catarino and Borges have presented the incomplete Leonardo numbers in [8]. Subsequently, the generalization of the Leonardo sequence has become a current issue. Shannon [35], Soykan [43], and Kuhapatanakul and Chobsorn [28] have followed different approaches to this issue. Shannon and Deveci have also studied the generalized and extended Leonardo sequences in [36]. Vieira et al. have introduced the two-dimensional and three-dimensional relations of the Leonardo sequence in [47, 49], respectively. Then, Vieira et al.

have examined the matrix formulas of Leonardo numbers in [48]. Soykan has studied the special cases of generalized Leonardo numbers in [44]. Karataş has introduced the complex Leonardo numbers in [23]. Mangueira et al. [31] have determined the Leonardo's bivariate and complex polynomials.

The quaternions formed with the Pauli matrices are called Pauli–quaternions [25]. Specifically, if the coefficients of these type quaternions are Fibonacci and Lucas numbers, respectively, the obtained quaternions are called the Pauli–Fibonacci quaternions and Pauli–Lucas quaternions [46].

In this study, we define the Pauli–Leonardo quaternions in the light of the related studies in the literature, especially [25] and [46]. Additionally, we give several fundamental and important formulas, properties, and equalities for the Pauli–Leonardo quaternions. We not only study the Pauli–Leonardo quaternions but also investigate some properties, including the relations between the Pauli–Leonardo quaternions, Pauli–Fibonacci quaternions, and Pauli–Lucas quaternions.

2 Preliminaries

This section provides some basic information about the Leonardo numbers and Pauli quaternions. First, some properties of the Leonardo numbers are given.

The characteristic equation of the recurrence relation (2) is $x^3 - 2x^2 + 1 = 0$, and Binet formula of the Leonardo numbers is given, for all $n \geq 0$, as

$$Le_n = \frac{2\xi^{n+1} - 2\delta^{n+1} - (\xi - \delta)}{\xi - \delta} \tag{3}$$

where $\xi = \frac{1+\sqrt{5}}{2}$ an $\delta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^3 - 2x^2 + 1 = 0$ [3,7].

For all $n \geq 0$, the Leonardo numbers satisfy the following equations given by [7]

$$Le_n = 2F_{n+1} - 1, \tag{4}$$

$$Le_n = \frac{2}{5}(L_{n+2} + L_n) - 1, \tag{5}$$

$$Le_{n+3} = \frac{1}{5}(L_{n+7} + L_{n+1}) - 1, \tag{6}$$

$$Le_n = L_{n+2} - F_{n+2} - 1, \tag{7}$$

and for all $n \geq 1$, the equations given by [3]

$$Le_{n-1} + Le_{n+1} = 2L_{n+1} - 2, \tag{8}$$

$$Le_n + 2F_n = Le_{n+1}, \tag{9}$$

$$Le_n + F_n + L_n = 2Le_n + 1. \tag{10}$$

The $(-n)$-th Leonardo number with negative subscript is defined as follows [3]:

$$Le_{-n} = (-1)^n (Le_{n-2} + 1) - 1, \text{ for all } n \geq 2. \tag{11}$$

Furthermore, the recurrence relation concerning negative subscripted Leonardo numbers is presented in [48] as:

$$Le_{-n} = -Le_{-n+1} + Le_{-n+2} - 1, \text{ for all } n > 0 \tag{12}$$

and some values of negative subscripted Leonardo numbers are: $Le_{-1} = -1, Le_{-2} = 1$, $Le_{-3} = -3, Le_{-4} = 3, Le_{-5} = -7$.

Additionally, Alp and Koçer [3] have presented the matrix representation of Leonardo numbers as a 3×3 matrix:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

and found some properties with respect to the matrix. Vieira et al. have studied the matrix formulas for Leonardo numbers in [48] for both non-negative and negative subscripts (see the studies [28, 43, 44] including matrix methods of the Leonardo numbers). We want to refer to also the studies [14, 22, 24, 30, 34, 37, 40–42, 50–53] including impressive techniques of matrix representations of some special numbers.

On the other hand, the quaternions introduced by Hamilton have various application areas and importance as the expansion of the complex numbers [17–19]. They are used in many areas, such as pure mathematics, applied mathematics, motion geometry, differential geometry, graph theory, computer animation, robotics, and others (cf. [1, 4, 9]). The algebra of quaternions is associative, non-commutative, and 4-dimensional Clifford algebra. The quaternion set is denoted by $\mathbb{H}$ and defined as: $\mathbb{H} = \{q | q = q_0 + q_1 i + q_2 j + q_3 k, q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ where i, j, k are the quaternionic units that satisfy the rules (for real quaternions): $i^2 = j^2 = k^2 = -1$, $ij = -ji = k, jk = -kj = i, ki = -ik = j$ [17–19].

The Pauli matrices determined by W. Pauli are Hermitian and unitary and given as follows [10,25,46]:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \boldsymbol{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \boldsymbol{\sigma_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \boldsymbol{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the following rules are held:

$$\begin{gathered} \boldsymbol{\sigma_1}^2 = \mathbf{1}, \boldsymbol{\sigma_2}^2 = \mathbf{1}, \boldsymbol{\sigma_3}^2 = \mathbf{1}, \\ \boldsymbol{\sigma_1}\boldsymbol{\sigma_2} = -\boldsymbol{\sigma_2}\boldsymbol{\sigma_1} = i\boldsymbol{\sigma_3}, \quad \boldsymbol{\sigma_2}\boldsymbol{\sigma_3} = -\boldsymbol{\sigma_3}\boldsymbol{\sigma_2} = i\boldsymbol{\sigma_1}, \quad \boldsymbol{\sigma_3}\boldsymbol{\sigma_1} = -\boldsymbol{\sigma_1}\boldsymbol{\sigma_3} = i\boldsymbol{\sigma_2}. \end{gathered} \tag{13}$$

The basis of Pauli quaternions is $\{\mathbf{1}, i\boldsymbol{\sigma_1}, i\boldsymbol{\sigma_2}, i\boldsymbol{\sigma_3}\}$ and the set is isomorphic to the set $\mathbb{H}$. Besides, an isomorphism exists from $\mathbb{H}$ to the set obtained via the map: $1 \to \mathbf{1}, i \to -i\boldsymbol{\sigma_1}$, $j \to -i\boldsymbol{\sigma_2}$, $k \to -i\boldsymbol{\sigma_3}$ or $1 \to \mathbf{1}$, $i \to i\boldsymbol{\sigma_1}$, $j \to i\boldsymbol{\sigma_2}, k \to i\boldsymbol{\sigma_3}$ [5, 6, 10, 25, 29, 46]. The Pauli matrices have wide application areas such as mathematics, physics, mathematical physics; see more detailed information in [5, 6, 10, 11, 16, 25, 29, 38, 46].

In [25], Kim has examined the Pauli quaternions as: $p = x_0\mathbf{1} + x_1\boldsymbol{\sigma_1} + x_2\boldsymbol{\sigma_2} + x_3\boldsymbol{\sigma_3}$. Then, Torunbalcı Aydın [46] has defined the Pauli–Fibonacci quaternions and Pauli–Lucas quaternions as follows, respectively:

$$Q_pF_n = F_n\mathbf{1} + F_{n+1}\boldsymbol{\sigma_1} + F_{n+2}\boldsymbol{\sigma_2} + F_{n+3}\boldsymbol{\sigma_3}, \tag{14}$$

$$Q_pL_n = L_n\mathbf{1} + L_{n+1}\boldsymbol{\sigma_1} + L_{n+2}\boldsymbol{\sigma_2} + L_{n+3}\boldsymbol{\sigma_3}. \tag{15}$$

For more detailed information about the Pauli quaternions, Pauli–Fibonacci quaternions, and Pauli–Lucas quaternions, we can refer to the studies [25, 46].

3 Pauli–Leonardo quaternions

In this section, we introduce the Pauli–Leonardo quaternions, and then we give some special formulas and properties of them.

Definition 3.1. *For all $n \geq 0$, the n^{th} Pauli–Leonardo quaternion is defined as follows:*

$$\mathcal{L}_n = Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3} \tag{16}$$

where Le_n is the n-th Leonardo number, and $\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{\sigma_3}$ satisfies the rules (13)*. The set of all Pauli–Leonardo quaternions is denoted by $\mathbb{L}$.*

Now, let us give the algebraic properties concerning the Pauli–Leonardo quaternions, such as equality, addition, subtraction, multiplication by a scalar, multiplication of any two Pauli–Leonardo quaternions, conjugate, norm, scalar and vector parts, respectively. Let us take $\mathcal{L}_n, \mathcal{L}_m \in \mathbb{L}$, then we have:

- **Equality:**

$$\mathcal{L}_n = \mathcal{L}_m \Leftrightarrow Le_n = Le_m, Le_{n+1} = Le_{m+1}, Le_{n+2} = Le_{m+2}, Le_{n+3} = Le_{m+3}$$

- **Addition/Subtraction:**

$$\begin{aligned}\mathcal{L}_n \pm \mathcal{L}_m = &(Le_n \pm Le_m)\,\mathbf{1} + (Le_{n+1} \pm Le_{m+1})\,\boldsymbol{\sigma_1} + (Le_{n+2} \pm Le_{m+2})\,\boldsymbol{\sigma_2} \\ &+ (Le_{n+3} \pm Le_{m+3})\,\boldsymbol{\sigma_3}\end{aligned}$$

- **Multiplication by a scalar:**

$$c\mathcal{L}_n = cLe_n\mathbf{1} + cLe_{n+1}\boldsymbol{\sigma_1} + cLe_{n+2}\boldsymbol{\sigma_2} + cLe_{n+3}\boldsymbol{\sigma_3},\ \ c \in \mathbb{R}$$

- **Multiplication:**

$$\begin{aligned}\mathcal{L}_n\mathcal{L}_m &= (Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3})\,(Le_m\mathbf{1} + Le_{m+1}\boldsymbol{\sigma_1} + Le_{m+2}\boldsymbol{\sigma_2} + Le_{m+3}\boldsymbol{\sigma_3}) \\ &= (Le_nLe_m + Le_{n+1}Le_{m+1} + Le_{n+2}Le_{m+2} + Le_{n+3}Le_{m+3})\,\mathbf{1} \\ &\quad + [(Le_nLe_{m+1} + Le_{n+1}Le_m) + i\,(Le_{n+2}Le_{m+3} - Le_{n+3}Le_{m+2})]\,\boldsymbol{\sigma_1} \\ &\quad + [(Le_nLe_{m+2} + Le_{n+2}Le_m) + i\,(Le_{n+3}Le_{m+1} - Le_{n+1}Le_{m+3})]\,\boldsymbol{\sigma_2} \\ &\quad + [(Le_nLe_{m+3} + Le_{n+3}Le_m) + i\,(Le_{n+1}Le_{m+2} - Le_{n+2}Le_{m+1})]\,\boldsymbol{\sigma_3}\end{aligned}$$

Also, the multiplication can be expressed as:

$$\mathcal{L}_n\mathcal{L}_m = \begin{bmatrix} Le_n & Le_{n+1} & Le_{n+2} & Le_{n+3} \\ Le_{n+1} & Le_n & -iLe_{n+3} & iLe_{n+2} \\ Le_{n+2} & iLe_{n+3} & Le_n & -iLe_{n+1} \\ Le_{n+3} & -iLe_{n+2} & iLe_{n+1} & Le_n \end{bmatrix} \begin{bmatrix} Le_m \\ Le_{m+1} \\ Le_{m+2} \\ Le_{m+3} \end{bmatrix}$$

- **Conjugate:**

$$\overline{\mathcal{L}}_n = Le_n\mathbf{1} - Le_{n+1}\boldsymbol{\sigma_1} - Le_{n+2}\boldsymbol{\sigma_2} - Le_{n+3}\boldsymbol{\sigma_3} \tag{17}$$

- **Norm:**

$$N\left(\mathcal{L}_n\right) = \sqrt{\left|\mathcal{L}_n\overline{\mathcal{L}}_n\right|} = \sqrt{\left|\overline{\mathcal{L}}_n\mathcal{L}_n\right|} = \sqrt{\left|Le_n^2 - Le_{n+1}^2 - Le_{n+2}^2 - Le_{n+3}^2\right|}$$

  Here, if $N(\mathcal{L}_n) = 1$, then $\mathcal{L}_n$ is called unit Pauli–Leonardo quaternion.

- **Scalar and Vector Parts:** The scalar part of $\mathcal{L}_n$ is represented as $S_{\mathcal{L}_n}$, and $S_{\mathcal{L}_n} = Le_n\mathbf{1}$. The vector part of $\mathcal{L}_n$ is represented by $V_{\mathcal{L}_n}$ and $V_{\mathcal{L}_n} = Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3}$.

  Therefore, $S_{\mathcal{L}_n \pm \mathcal{L}_m} = S_{\mathcal{L}_n} \pm S_{\mathcal{L}_m} = \left(\mathcal{L}_n \pm \mathcal{L}_m\right)\mathbf{1}$ and $V_{\mathcal{L}_n \pm \mathcal{L}_m} = V_{\mathcal{L}_n} \pm V_{\mathcal{L}_m}$.

Theorem 3.1 (Recurrence Relation). *Let $\mathcal{L}_n$ be n-th Pauli–Leonardo quaternion. For all $n \geq 2$, the following recurrence relation holds:*

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + W \tag{18}$$

where $W = \mathbf{1} + \boldsymbol{\sigma_1} + \boldsymbol{\sigma_2} + \boldsymbol{\sigma_3}$. Additionally, for all $n \geq 2$ the following recurrence relation is satisfied for Pauli–Leonardo quaternions:

$$\mathcal{L}_{n+1} = 2\mathcal{L}_n - \mathcal{L}_{n-2}. \tag{19}$$

Proof. By using the equations (1) and (16), the following can be written:

$$\begin{aligned}\mathcal{L}_n =& Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3} \\ =& \left(Le_{n-1} + Le_{n-2} + 1\right)\mathbf{1} + \left(Le_n + Le_{n-1} + 1\right)\boldsymbol{\sigma_1} + \left(Le_{n+1} + Le_n + 1\right)\boldsymbol{\sigma_2} \\ &+ \left(Le_{n+2} + Le_{n+1} + 1\right)\boldsymbol{\sigma_3} \\ &= Le_{n-1}\mathbf{1} + Le_n\boldsymbol{\sigma_1} + Le_{n+1}\boldsymbol{\sigma_2} + Le_{n+2}\boldsymbol{\sigma_3} + Le_{n-2}\mathbf{1} + Le_{n-1}\boldsymbol{\sigma_1} + Le_n\boldsymbol{\sigma_2} \\ &+ Le_{n+1}\boldsymbol{\sigma_3} + \mathbf{1} + \boldsymbol{\sigma_1} + \boldsymbol{\sigma_2} + \boldsymbol{\sigma_3} \\ =& \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + W\end{aligned}$$

where we use the expression $W = \mathbf{1} + \boldsymbol{\sigma_1} + \boldsymbol{\sigma_2} + \boldsymbol{\sigma_3}$ for the sake of brevity. The other recurrence relation can be proved by using (2). □

We can also define the negative subscripted Pauli–Leonardo quaternions as follows:

$$\mathcal{L}_{-n} = Le_{-n}\mathbf{1} + Le_{-n+1}\boldsymbol{\sigma_1} + Le_{-n+2}\boldsymbol{\sigma_2} + Le_{-n+3}\boldsymbol{\sigma_3}, \text{ for all } n > 0. \tag{20}$$

Theorem 3.2. *Let $\mathcal{L}_{-n}$ be the $(-n)$-th Pauli–Leonardo quaternion and Le_n be the n-th Leonardo number, respectively. The following equations are valid.*

$$\begin{aligned}\mathcal{L}_{-n} &= (-1)^n \left[(Le_{n-2}+1)\,\mathbf{1} + (Le_{n-4}+1)\,\boldsymbol{\sigma_2}\right] \\ &\quad + (-1)^{n-1}\left[(Le_{n-3}+1)\,\boldsymbol{\sigma_1} + (Le_{n-5}+1)\,\boldsymbol{\sigma_3}\right] - W, \textit{for all } n \geq 2\end{aligned} \tag{21}$$

$$\mathcal{L}_{-n} = -\mathcal{L}_{-n+1} + \mathcal{L}_{-n+2} - W, \textit{for all } n > 0 \tag{22}$$

Proof. By using the equations (11) (for equation (21)), (12) (for equation (22)) and (20), the proof can be completed. □

In the following Table 2 and Table 3, we construct numerical algorithms in order to calculate the n-th and $(n+1)$-th terms of the Pauli–Leonardo quaternions.

Table 2. Numerical Algorithm 1

A Numerical Algorithm for Finding n-th Term of the Pauli–Leonardo Quaternion
1. Begin
2. Input $\mathcal{L}_0, \mathcal{L}_1$ and W
3. Compose $\mathcal{L}_n$ according to the equation (18) for all $n \geq 2$
4. Count up $\mathcal{L}_n$
5. Output $\mathcal{L}_n = Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3}$
6. Finish

Table 3. Numerical Algorithm 2

A Numerical Algorithm for Finding $(n+1)$-th Term of the Pauli–Leonardo Quaternion
1. Begin
2. Input $\mathcal{L}_0$ and $\mathcal{L}_1$
3. Compose $\mathcal{L}_n$ according to the equation (19) for all $n \geq 2$
4. Count up $\mathcal{L}_{n+1}$
5. Output $\mathcal{L}_{n+1} = Le_{n+1}\mathbf{1} + Le_{n+2}\boldsymbol{\sigma_1} + Le_{n+3}\boldsymbol{\sigma_2} + Le_{n+4}\boldsymbol{\sigma_3}$
6. Finish

In the following Table 4, we also form a numerical algorithm for calculating the $(-n)^{th}$ term of the Pauli–Leonardo quaternion.

Table 4. Numerical Algorithm 3

A Numerical Algorithm for Finding $(-n)$-th Term of the Pauli–Leonardo Quaternion
1. Begin
2. Input $\mathcal{L}_{-1}, \mathcal{L}_0$ and W
3. Compose $\mathcal{L}_{-n}$ according to the equation (22) for all $n > 0$
4. Count up $\mathcal{L}_{-n}$
5. Output $\mathcal{L}_{-n} = Le_{-n}\mathbf{1} + Le_{-n+1}\boldsymbol{\sigma_1} + Le_{-n+2}\boldsymbol{\sigma_2} + Le_{-n+3}\boldsymbol{\sigma_3}$
6. Finish

It should be noted that we are interested in the non-negative Pauli–Leonardo quaternions in general terms throughout this paper. We also give the definition of negative subscripted Pauli–Leonardo quaternions (20), and equations (21) and (22). The following equations, formulas, and properties can be expressed separately for negative subscripted Pauli–Leonardo quaternions.

Theorem 3.3. *Let $\mathcal{L}_n$ be the n^{th} Pauli–Leonardo quaternion. For all $n \geq 0$, the following relations exist.*

(i) $\mathcal{L}_n + \overline{\mathcal{L}}_n = 2Le_n\mathbf{1}$

(ii) $\mathcal{L}_n^2 = 2Le_n\mathcal{L}_n\mathbf{1} - \mathcal{L}_n\overline{\mathcal{L}}_n$

(iii) $\mathcal{L}_n\mathbf{1} - \mathcal{L}_{n+1}\boldsymbol{\sigma_1} - \mathcal{L}_{n+2}\boldsymbol{\sigma_2} - \mathcal{L}_{n+3}\boldsymbol{\sigma_3} = (Le_n - Le_{n+2} - Le_{n+4} - Le_{n+6})\,\mathbf{1}$

Proof. (i) By using the (16) and (17), we can establish:

$$\begin{aligned}\mathcal{L}_n + \overline{\mathcal{L}}_n &= Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3} + Le_n\mathbf{1} - Le_{n+1}\boldsymbol{\sigma_1} \\ &\quad - Le_{n+2}\boldsymbol{\sigma_2} - Le_{n+3}\boldsymbol{\sigma_3} \\ &= 2Le_n\mathbf{1}.\end{aligned}$$

(ii) By means of the equation which is seen in part (i), we get:

$$\mathcal{L}_n\mathcal{L}_n = \mathcal{L}_n\left(2Le_n\mathbf{1} - \overline{\mathcal{L}}_n\right) \Rightarrow 2Le_n\mathcal{L}_n\mathbf{1} - \mathcal{L}_n\overline{\mathcal{L}}_n.$$

(iii) Via (16) and (13), we have:

$$\begin{aligned}&\mathcal{L}_n\mathbf{1} - \mathcal{L}_{n+1}\boldsymbol{\sigma_1} - \mathcal{L}_{n+2}\boldsymbol{\sigma_2} - \mathcal{L}_{n+3}\boldsymbol{\sigma_3} \\ &\qquad = (Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3})\,\mathbf{1} \\ &\qquad\quad - (Le_{n+1}\mathbf{1} + Le_{n+2}\boldsymbol{\sigma_1} + Le_{n+3}\boldsymbol{\sigma_2} + Le_{n+4}\boldsymbol{\sigma_3})\,\boldsymbol{\sigma_1} \\ &\qquad\quad - (Le_{n+2}\mathbf{1} + Le_{n+3}\boldsymbol{\sigma_1} + Le_{n+4}\boldsymbol{\sigma_2} + Le_{n+5}\boldsymbol{\sigma_3})\,\boldsymbol{\sigma_2} \\ &\qquad\quad - (Le_{n+3}\mathbf{1} + Le_{n+4}\boldsymbol{\sigma_1} + Le_{n+5}\boldsymbol{\sigma_2} + Le_{n+6}\boldsymbol{\sigma_3})\,\boldsymbol{\sigma_3} \\ &\qquad = (Le_n - Le_{n+2} - Le_{n+4} - Le_{n+6})\,\mathbf{1}.\end{aligned}$$

□

Theorem 3.4. *Let $\mathcal{L}_n$ and Q_pF_n be the n^{th} Pauli–Leonardo quaternion and Pauli–Fibonacci quaternion respectively. The following relations are satisfied.*

(i) $\mathcal{L}_n = 2Q_pF_{n+1} - W,$ *for all* $n \geq 0$

(ii) $\mathcal{L}_{n+1} = \mathcal{L}_n + 2Q_pF_n,$ *for all* $n \geq 1$

Proof. By using (16), (4) and (14), we have:

$$\begin{aligned}\mathcal{L}_n &= Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3} \\ &= (2F_{n+1} - 1)\,\mathbf{1} + (2F_{n+2} - 1)\,\boldsymbol{\sigma_1} + (2F_{n+3} - 1)\,\boldsymbol{\sigma_2} + (2F_{n+4} - 1)\,\boldsymbol{\sigma_3} \\ &= 2\,(F_{n+1}\mathbf{1} + F_{n+2}\boldsymbol{\sigma_1} + F_{n+3}\boldsymbol{\sigma_2} + F_{n+4}\boldsymbol{\sigma_3}) - (\mathbf{1} + \boldsymbol{\sigma_1} + \boldsymbol{\sigma_2} + \boldsymbol{\sigma_3}) \\ &= 2Q_pF_{n+1} - W.\end{aligned}$$

The other property can be shown easily by using the equation (9). □

Theorem 3.5. *Let $\mathcal{L}_n$ and Q_pL_n be the n^{th} Pauli–Leonardo quaternion and Pauli–Lucas quaternion, respectively. The following equalities are satisfied.*

(i) $\mathcal{L}_n = \frac{2}{5}(Q_pL_{n+2} + Q_pL_n) - W, \textit{for all}\, n \geq 0$

(ii) $\mathcal{L}_{n+3} = \frac{1}{5}(Q_pL_{n+7} + Q_pL_{n+1}) - W, \textit{for all}\, n \geq 0$

(iii) $\mathcal{L}_{n-1} + \mathcal{L}_{n+1} = 2Q_pL_{n+1} - 2W, \textit{for all}\, n \geq 1$

Proof. Via utilizing the equations (5), (6), (8), (15) and (16), we can complete the proof. □

Theorem 3.6. *Let $\mathcal{L}_n, Q_pF_n$ and Q_pL_n be the n^{th} Pauli–Leonardo quaternion, Pauli–Fibonacci quaternion and Pauli–Lucas quaternion, respectively. The following equalities are held.*

(i) $\mathcal{L}_n = Q_pL_{n+2} - Q_pF_{n+2} - W, \textit{for all}\, n \geq 0$

(ii) $Q_pF_n + Q_pL_n = \mathcal{L}_n + W, \textit{for all}\, n \geq 1$

Proof. We can complete the proof easily by using (7), (10), (14), (15) and (16). □

Theorem 3.7 (**Binet Formula**)**.** *Let $\mathcal{L}_n$ be n-th Pauli–Leonardo quaternion. For all $n \geq 0$, the following Binet formula is satisfied for Pauli–Leonardo quaternions:*

$$\mathcal{L}_n = 2\left(\frac{\xi^{n+1}\breve{\xi} - \delta^{n+1}\breve{\delta}}{\xi - \delta}\right) - W$$

where

$$\begin{cases} \breve{\xi} = \mathbf{1} + \xi\boldsymbol{\sigma_1} + \xi^2\boldsymbol{\sigma_2} + \xi^3\boldsymbol{\sigma_3}, \\ \breve{\delta} = \mathbf{1} + \delta\boldsymbol{\sigma_1} + \delta^2\boldsymbol{\sigma_2} + \delta^3\boldsymbol{\sigma_3}. \end{cases}$$

Proof. By utilizing the definition of Pauli–Leonardo quaternion (16) and the Binet formula of the Leonardo numbers (3), we get:

$$\begin{aligned}
\mathcal{L}_n &= Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3} \\
&= \left[2\left(\frac{\xi^{n+1} - \delta^{n+1}}{\xi - \delta}\right) - 1\right]\mathbf{1} + \left[2\left(\frac{\xi^{n+2} - \delta^{n+2}}{\xi - \delta}\right) - 1\right]\boldsymbol{\sigma_1} \\
&\quad + \left[2\left(\frac{\xi^{n+3} - \delta^{n+3}}{\xi - \delta}\right) - 1\right]\boldsymbol{\sigma_2} + \left[2\left(\frac{\xi^{n+4} - \delta^{n+4}}{\xi - \delta}\right) - 1\right]\boldsymbol{\sigma_3} \\
&= 2\left[\frac{\xi^{n+1}\left(\mathbf{1} + \xi\boldsymbol{\sigma_1} + \xi^2\boldsymbol{\sigma_2} + \xi^3\boldsymbol{\sigma_3}\right) - \delta^{n+1}\left(\mathbf{1} + \delta\boldsymbol{\sigma_1} + \delta^2\boldsymbol{\sigma_2} + \delta^3\boldsymbol{\sigma_3}\right)}{\xi - \delta}\right] - W \\
&= 2\left(\frac{\xi^{n+1}\breve{\xi} - \delta^{n+1}\breve{\delta}}{\xi - \delta}\right) - W.
\end{aligned}$$

□

Theorem 3.8 (Generating Function). *Let $\mathcal{L}_n$ be n-th Pauli–Leonardo quaternion. For all $n \geq 0$, the generating function for Pauli–Leonardo quaternions is written as follows:*

$$\sum_{n=0}^{\infty} \mathcal{L}_n = \frac{\mathcal{L}_0 + (\mathcal{L}_1 - 2\mathcal{L}_0)\,x + (\mathcal{L}_2 - 2\mathcal{L}_1)\,x^2}{1 - 2x + x^3}.$$

Proof. Assume that the following equality is the generating function of the Pauli–Leonardo quaternions.

$$\sum_{n=0}^{\infty} \mathcal{L}_n x^n = \mathcal{L}_0 + \mathcal{L}_1 x + \mathcal{L}_2 x^2 + \cdots + \mathcal{L}_n x^n + \cdots$$

By using the equation (19), it can be also written as:

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{L}_n x^n &= \mathcal{L}_0 + \mathcal{L}_1 x + \mathcal{L}_2 x^2 + \sum_{n=3}^{\infty} \mathcal{L}_n x^n \\
&= \mathcal{L}_0 + \mathcal{L}_1 x + \mathcal{L}_2 x^2 + \sum_{n=3}^{\infty} (2\mathcal{L}_{n-1} - \mathcal{L}_{n-3})\,x^n \\
&= \mathcal{L}_0 + \mathcal{L}_1 x + \mathcal{L}_2 x^2 + \sum_{n=3}^{\infty} 2\mathcal{L}_{n-1} x^n - \sum_{n=3}^{\infty} \mathcal{L}_{n-3} x^n \\
&= \mathcal{L}_0 + \mathcal{L}_1 x + \mathcal{L}_2 x^2 + 2\sum_{n=2}^{\infty} \mathcal{L}_n x^{n+1} - \sum_{n=0}^{\infty} \mathcal{L}_n x^{n+3} \\
&= \mathcal{L}_0 + \mathcal{L}_1 x + \mathcal{L}_2 x^2 + 2x\sum_{n=2}^{\infty} \mathcal{L}_n x^n - x^3\sum_{n=0}^{\infty} \mathcal{L}_n x^n \\
&= \mathcal{L}_0 + \mathcal{L}_1 x + \mathcal{L}_2 x^2 + 2x\left(\sum_{n=0}^{\infty} \mathcal{L}_n x^n - \mathcal{L}_0 - \mathcal{L}_1 x\right) - x^3\sum_{n=0}^{\infty} \mathcal{L}_n x^n.
\end{aligned}$$

Then, we have:

$$\sum_{n=0}^{\infty} \mathcal{L}_n = \frac{\mathcal{L}_0 + (\mathcal{L}_1 - 2\mathcal{L}_0)\,x + (\mathcal{L}_2 - 2\mathcal{L}_1)\,x^2}{1 - 2x + x^3}. \qquad \square$$

Theorem 3.9 (Exponential Generating Function). *Let $\mathcal{L}_n$ be the n-th Pauli–Leonardo quaternion. For all $n \geq 0$, the exponential generating function for Pauli–Leonardo quaternions is given as:*

$$\sum_{n=0}^{\infty} \mathcal{L}_n \frac{y^n}{n!} = 2\left(\frac{\xi\breve{\xi}e^{\xi y} - \delta\breve{\delta}e^{\delta y}}{\xi - \delta}\right) - We^y.$$

Proof. By means of the equation (3.7), we get:

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{L}_n \frac{y^n}{n!} &= \sum_{n=0}^{\infty} \left[2\left(\frac{\xi^{n+1}\breve{\xi} - \delta^{n+1}\breve{\delta}}{\xi - \delta}\right) - W\right]\frac{y^n}{n!} \\
&= \frac{2\xi\breve{\xi}}{\xi - \delta}\sum_{n=0}^{\infty} \frac{(\xi y)^n}{n!} - \frac{2\delta\breve{\delta}}{\xi - \delta}\sum_{n=0}^{\infty} \frac{(\delta y)^n}{n!} - W\sum_{n=0}^{\infty} \frac{y^n}{n!} \\
&= 2\left(\frac{\xi\breve{\xi}e^{\xi y} - \delta\breve{\delta}e^{\delta y}}{\xi - \delta}\right) - We^y.
\end{aligned} \qquad \square$$

Thanks to the summation formulas for the Leonardo numbers given in [7], we get the following summation formulas for our new type of particular number system titled the Pauli–Leonardo quaternions (see also the studies [28, 43, 44] which include the summation formulas with respect to the Leonardo numbers).

Theorem 3.10. *Let $\mathcal{L}_n$ be the n-th Pauli–Leonardo quaternion. For all $m, n \geq 0$, the following summation formulas are satisfied.*

(i) $\sum\limits_{n=0}^{m} \mathcal{L}_n = \mathcal{L}_{m+2} - (mW + 2\mathbf{1} + 4\boldsymbol{\sigma_1} + 6\boldsymbol{\sigma_2} + 10\boldsymbol{\sigma_3})$

(ii) $\sum\limits_{n=0}^{m} \mathcal{L}_{2n} = \mathcal{L}_{2m+1} - [mW + 2(\boldsymbol{\sigma_1} + \boldsymbol{\sigma_2} + 2\boldsymbol{\sigma_3})]$

(iii) $\sum\limits_{n=0}^{m} \mathcal{L}_{2n+1} = \mathcal{L}_{2m+2} - (mW + 2\boldsymbol{\sigma_1} + 4\boldsymbol{\sigma_2} + 6\boldsymbol{\sigma_3})$

Proof. (i) By using the definition of Pauli–Leonardo quaternions (16), we have:

$$\begin{aligned}\sum_{n=0}^{m} \mathcal{L}_n &= \sum_{n=0}^{m} (Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3}) \\ &= \sum_{n=0}^{m} Le_n\mathbf{1} + \sum_{n=0}^{m} Le_{n+1}\boldsymbol{\sigma_1} + \sum_{n=0}^{m} Le_{n+2}\boldsymbol{\sigma_2} + \sum_{n=0}^{m} Le_{n+3}\boldsymbol{\sigma_3}\end{aligned}$$

where the following sum formula is given in [7]

$$\sum_{n=0}^{m} Le_n = Le_{m+2} - (m+2)\,. \tag{23}$$

Also, by using (23), we obtain $\sum\limits_{n=0}^{m} Le_{n+1}$, $\sum\limits_{n=0}^{m} Le_{n+2}$ and $\sum\limits_{n=0}^{m} Le_{n+3}$.

Then, we have:

$$\begin{aligned}\sum_{n=0}^{m} \mathcal{L}_n &= [Le_{m+2} - (m+2)]\,\mathbf{1} + [Le_{m+3} - (m+4)]\,\boldsymbol{\sigma_1} + [Le_{m+4} - (m+6)]\,\boldsymbol{\sigma_2} \\ &\quad + [Le_{m+5} - (m+10)]\,\boldsymbol{\sigma_3} \\ &= Le_{m+2}\mathbf{1} + Le_{m+3}\boldsymbol{\sigma_1} + Le_{m+4}\boldsymbol{\sigma_2} + Le_{m+5}\boldsymbol{\sigma_3} \\ &\quad - [(m+2)\mathbf{1} + (m+4)\,\boldsymbol{\sigma_1} + (m+6)\,\boldsymbol{\sigma_2} + (m+10)\,\boldsymbol{\sigma_3}] \\ &= \mathcal{L}_{m+2} - [(m+2)\mathbf{1} + (m+4)\,\boldsymbol{\sigma_1} + (m+6)\,\boldsymbol{\sigma_2} + (m+10)\,\boldsymbol{\sigma_3}] \\ &= \mathcal{L}_{m+2} - (mW + 2\mathbf{1} + 4\boldsymbol{\sigma_1} + 6\boldsymbol{\sigma_2} + 10\boldsymbol{\sigma_3})\,.\end{aligned}$$

The other parts can be obtained in the same manner. □

Inspired by the studies [3, 48], we also establish the matrix equalities for Pauli–Leonardo quaternions in the following Theorem 3.11.

Theorem 3.11. *Let $\mathcal{L}_n$ be the n-th Pauli–Leonardo quaternion. For all $n > 0$, the following matrix equalities are held.*

$$\begin{pmatrix} \mathcal{L}_3 & \mathcal{L}_2 & \mathcal{L}_1 \\ \mathcal{L}_2 & \mathcal{L}_1 & \mathcal{L}_0 \\ \mathcal{L}_1 & \mathcal{L}_0 & \mathcal{L}_{-1} \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}^n = \begin{pmatrix} \mathcal{L}_{n+3} & \mathcal{L}_{n+2} & \mathcal{L}_{n+1} \\ \mathcal{L}_{n+2} & \mathcal{L}_{n+1} & \mathcal{L}_n \\ \mathcal{L}_{n+1} & \mathcal{L}_n & \mathcal{L}_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{L}_0 & \mathcal{L}_1 & \mathcal{L}_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}^n = \begin{pmatrix} \mathcal{L}_n & \mathcal{L}_{n+1} & \mathcal{L}_{n+2} \end{pmatrix}$$

Proof. The proof is obvious by the mathematical induction on n, so we omit the proof. □

Now, the $\mathbb{R}$–linear transformations which representing the left and right multiplications in $\mathbb{L}$ via utilizing de Moivre's formula are presented by following the same manner in the studies [6, 25, 46].

Consider $\mathcal{L}_n, \mathcal{L}_m \in \mathbb{L}$, then the followings are given:

$$\begin{aligned} \phi_{L_{\mathcal{L}_n}} &: \mathbb{L} \to \mathbb{L} \\ &\quad \mathcal{L}_m \to \phi_{L_{\mathcal{L}_n}}(\mathcal{L}_m) = \mathcal{L}_n\mathcal{L}_m \end{aligned}$$

where $\phi_{L_{\mathcal{L}_n}}$ is written as

$$A_{\phi_{L_{\mathcal{L}_n}}} = \begin{pmatrix} Le_n & Le_{n+1} & Le_{n+2} & Le_{n+3} \\ Le_{n+1} & Le_n & -iLe_{n+3} & iLe_{n+2} \\ Le_{n+2} & iLe_{n+3} & Le_n & -iLe_{n+1} \\ Le_{n+3} & -iLe_{n+2} & iLe_{n+1} & Le_n \end{pmatrix}$$

and

$$\begin{aligned} \phi_{R_{\mathcal{L}_n}} &: \mathbb{L} \to \mathbb{L} \\ &\quad \mathcal{L}_m \to \phi_{R_{\mathcal{L}_n}}(\mathcal{L}_m) = \mathcal{L}_m\mathcal{L}_n \end{aligned}$$

where $\phi_{R_{\mathcal{L}_n}}$ is also written as

$$A_{\phi_{R_{\mathcal{L}_n}}} = \begin{pmatrix} Le_n & Le_{n+1} & Le_{n+2} & Le_{n+3} \\ Le_{n+1} & Le_n & iLe_{n+3} & -iLe_{n+2} \\ Le_{n+2} & -iLe_{n+3} & Le_n & iLe_{n+1} \\ Le_{n+3} & iLe_{n+2} & -iLe_{n+1} & Le_n \end{pmatrix}.$$

For the unit Pauli–Leonardo quaternion $\mathcal{L}_n$, the mapping $\phi_{\mathcal{L}_n} : \mathbb{L} \to \mathbb{L}$ is defined as $\phi_{\mathcal{L}_n} = \phi_{L_{\mathcal{L}_n}} \circ \phi_{R_{\mathcal{L}_n}}$, c is a real number and $\phi_{\mathcal{L}_n} = \phi_{R_{\mathcal{L}_n}} \circ \phi_{L_{\mathcal{L}_n}}$, and also $\phi_{L_{\mathcal{L}_n}}$ and $\phi_{R_{\mathcal{L}_n}}$ are operators, which are identified as in $A_{\phi_{L_{\mathcal{L}_n}}}$ and $A_{\phi_{R_{\mathcal{L}_n}}}$.

Also, for all $\mathcal{L}_n, \mathcal{L}_m, \mathcal{L}_k \in \mathbb{L}$ and $c \in \mathbb{R}$, the listed equalities are satisfied.

(i) $\mathcal{L}_n = \mathcal{L}_m$ if and only if $\phi_{L_{\mathcal{L}_n}}(\mathcal{L}_m) = \phi_{L_{\mathcal{L}_n}}(\mathcal{L}_k)$ and $\phi_{R_{\mathcal{L}_n}}(\mathcal{L}_m) = \phi_{R_{\mathcal{L}_n}}(\mathcal{L}_k)$

(ii) $\phi_{L_{\mathcal{L}_n}}(\mathcal{L}_m + \mathcal{L}_k) = \phi_{L_{\mathcal{L}_n}}(\mathcal{L}_m) + \phi_{L_{\mathcal{L}_n}}(\mathcal{L}_k)$ and $\phi_{R_{\mathcal{L}_n}}(\mathcal{L}_m + \mathcal{L}_k) = \phi_{R_{\mathcal{L}_n}}(\mathcal{L}_m) + \phi_{R_{\mathcal{L}_n}}(\mathcal{L}_k)$

(iii) $\phi_{L_{\mathcal{L}_n}}(c\mathcal{L}_m) = c\phi_{L_{\mathcal{L}_n}}(\mathcal{L}_m)$ and $\phi_{R_{\mathcal{L}_n}}(c\mathcal{L}_m) = c\phi_{R_{\mathcal{L}_n}}(\mathcal{L}_m)$

(iv) $\phi_{L_{\mathcal{L}_n}}(\mathcal{L}_m)\phi_{R_{\mathcal{L}_n}}(\mathcal{L}_m) = \phi_{R_{\mathcal{L}_n}}(\mathcal{L}_m)\phi_{L_{\mathcal{L}_n}}(\mathcal{L}_m)$

In addition to these, the mappings of ϑ_L and ϑ_R which are obtained as follows:

$$\vartheta_L : (\mathbb{L}, +, \cdot) \to (M_{(4,\mathbb{R})}, \oplus, \otimes)$$

$$\vartheta_L(Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3}) = \begin{pmatrix} Le_n & Le_{n+1} & Le_{n+2} & Le_{n+3} \\ Le_{n+1} & Le_n & -iLe_{n+3} & iLe_{n+2} \\ Le_{n+2} & iLe_{n+3} & Le_n & -iLe_{n+1} \\ Le_{n+3} & -iLe_{n+2} & iLe_{n+1} & Le_n \end{pmatrix}$$

and

$$\vartheta_R : (\mathbb{L}, +, \cdot) \to (M_{(4,\mathbb{R})}, \oplus, \otimes)$$

$$\vartheta_R(Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3}) = \begin{pmatrix} Le_n & Le_{n+1} & Le_{n+2} & Le_{n+3} \\ Le_{n+1} & Le_n & iLe_{n+3} & -iLe_{n+2} \\ Le_{n+2} & -iLe_{n+3} & Le_n & iLe_{n+1} \\ Le_{n+3} & iLe_{n+2} & -iLe_{n+1} & Le_n \end{pmatrix}$$

are isomorphisms. Besides, ϑ_L and ϑ_R are bijective $\vartheta_L(\mathcal{L}_n\mathcal{L}_m) = \vartheta_L(\mathcal{L}_n)\vartheta_L(\mathcal{L}_m)$ and $\vartheta_R(\mathcal{L}_n\mathcal{L}_m) = \vartheta_R(\mathcal{L}_n)\vartheta_R(\mathcal{L}_m)$.

4 Conclusions

In this paper, we have investigated a new type of number system, which is named a system of Pauli–Leonardo quaternions. We scrutinize some special formulas and equalities concerning them. Considering the Pauli–Fibonacci and Pauli–Lucas quaternions introduced in [46], we have obtained the relations between these quaternions and the Pauli–Fibonacci quaternions. We have given the recurrence relation, Binet formula, generating function, exponential generating function, matrix formulas, and summation formulas for the newly defined number system. Moreover, we have constructed some algorithms for finding the terms of the Pauli–Leonardo quaternions. Also, we have presented the matrix representation and $\mathbb{R}$-linear transformation for them.

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