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THE MIXED BOUNDARY VALUE PROBLEM FOR A THIRD ORDER EQUATION WITH MULTIPLE CHARACTERISTICS

Yu.P.Apakov
Namangan Engineering-Pedagogical Institute
8, Ziyokor str., Namangan 160103, Uzbekistan
E-mail: apakov.1956@mail.ru

ABSTRACT

In the paper, the boundary value problem is considered for equation $Uxxx - Uyy = 0$ *in the domain* $D = \{(x;\ y);\ 0 < x < a\ ;\ 0 < y < b\}$. *Uniqueness of the stated problem is proved by the method of energy integral. The solution is constructed by the Fourier method. Eigenvalues and eigenfunctions are found for a problem of Sturm-Louville's type.*

Key words: *mixed boundary value problem, multiple characteristics, method of energy integral.*

ÖZET

Bu makalede, $D = \{(x;y);0 < x < a;0 < y < b\}$ *bölgesinde* $U_{xxx} - U_{yy} = 0$ *eşitliği için sınır değer problemi incelenmiştir. Ortaya konulan problemin tekliği enerji integrali metoduyla ispatlanmıştır. Bu çözüm Fourier metoduyla kurulmuştur. Özdeğerler ve özfonksiyonlar Sturn-Louville tipli bir problem için bulunmuştur.*

Anahtar Kelimeler: *karışık sınır değer problemi, çoklu karakteristikler, enerji integralinin metodu.*

1. Introduction

Consider the equation

$$Uxxx - Uyy = 0 \qquad (1)$$

in the domain $D = \{(x;\ y);\ 0 < x < a\ ;\ 0 < y < b\}$.

First works devoted to the equation (1) were papers of Italian mathematics H. Block [6] and E. Del Vecchio [12,13]. Then their results were generalized in the paper by L. Cattabriga [7] where he constructed fundamental solutions and developed the theory of potentials. Later, various boundary value problems were studied in [1]-[2] using fundamental solutions constructed in [7].

Some local boundary value problems for the equation (1) were constructed in [3]-[5] where solutions were constructed using the Fourier method.

2. Statement of the problem

We study the following boundary value problem for the equation (1) in the domain *D*.

Problem A_α. *To find a regular solution $U(x,y)\in C^{3,2}_{x,y}(D)\cap C^{2,1}_{x,y}(\overline{D})$ of the equation (1) in the domain D satisfying the boundary conditions*

$$\left.\begin{aligned}\alpha U(x,0)+\beta U_y(x,0)=0,\\ \gamma U(x,b)+\delta U_y(x,b)=0,\end{aligned}\right\}\ 0<x<a, \qquad (2)$$

$$U_{xx}(0,y)=\varphi_1(y),\quad U_x(a;y)=\varphi_2(y),\quad U_{xx}(a,y)=\varphi_3(y),\quad 0\le y\le b \qquad (3)$$

where $\alpha,\beta,\gamma,\delta$ *are constants such that* $\alpha^2+\beta^2\neq 0,\ \gamma^2+\delta^2\neq 0$ *and functions* $\varphi_j\in C^1[0,b],\ j=1,3,\ \varphi_2\in C^2[0,b],\ \varphi_i(0)=\varphi_i(b), i=1,2,3.$

Note that Problem A_α was considered at $\alpha=\gamma=1,\ \beta=\delta=0$ [3] at $\beta=\delta=1, \alpha=\gamma=0$ in [4], and an analogous problem was considered in [5].

3. Uniqueness of the solution

Theorem 1. If $\alpha\beta\le 0,\ \gamma\delta\ge 0,$ then the homogeneous problem A_α has not more than one solution.

Proof. Suppose the opposite, i.e. let $U_1(x,y)$ and $U_2(x,y)$ be solutions of Problem A_α . Then $U(x,y)=U_1(x,y)-U_2(x,y)$ is the solution of the homogeneous problem.

Consider the identity

$$\frac{\partial}{\partial x}\left(UU_{xx}-\frac{1}{2}U_x^2\right)-\frac{\partial}{\partial y}\left(UU_y\right)+U_y^2=0.$$

Integrating it in D and taking into account homogeneous boundary conditions, we obtain

$$\frac{1}{2}\int_0^b U_x^2(0,y)dy-\int_0^a U(x,b)U_y(x,b)dx+\int_0^a U(x,0)U_y(x,0)dx+\iint_D U_y^2(x,y)dxdy=0\cdot$$

Requiring $\alpha\neq 0,\ \gamma\neq 0$ in (2), we have

$$\frac{1}{2}\int_0^b U_x^2(0,y)dy-\frac{\delta}{\gamma}\int_0^a U_y^2(x,b)dx-\frac{\beta}{\alpha}\int_0^a U_y^2(x,0)dx+\iint_D U_y^2(x,y)dxdy=0.$$

Taking into account conditions of theorem, we obtain $U_y(x,y)=0$, i.e. $U(x,y)=f(x)$. $U_y(x,0)=0$ therefore $U(x,0)=0$, hence, $f(x)\equiv 0$ or $U(x,y)=0$. If $\alpha\neq 0,$ $\delta\neq 0,\ \beta\neq 0,\ \gamma\neq 0,$ then we also have $U(x,y)=0$.

4. Existence of the solution

Consider the following subsidiary problem: to find a non-zero solution of the equation (1) satisfying conditions (2) which is represented in the form

$$U(x,y)=X(x)Y(y). \tag{4}$$

Substituting (4) in (1) and separating the variables, we obtain

$$Y''+\lambda Y=0, \tag{5}$$

$$X'''+\lambda X=0. \tag{6}$$

We have from (5) and (2) the problem of Sturm-Louville's type:

$$\left.\begin{array}{l} Y''+\lambda Y=0,\\ \alpha Y(0)+\beta Y'(0),\\ \gamma(b)+\delta Y'(b). \end{array}\right\} \tag{7}$$

It is known [10] that eigenvalues of the parameter λ¸ for the problem (7) exist only at $\lambda > 0$, the corresponding general solution has the form

$$Y(y) = C_1 \cos\sqrt{\lambda} y + C_2 \sin\sqrt{\lambda} y$$

where C_1, C_2 are arbitrary constants.

Satisfying the conditions of the problem (7), we obtain the transcendental equation for determination of λ:

$$ctg\sqrt{\lambda} y = \frac{\alpha\gamma + \lambda\delta\beta}{\sqrt{\lambda}(\gamma\beta - \alpha\delta)}. \tag{8}$$

Putting $\xi = \sqrt{\lambda} b$, we have

$$ctg\xi = \frac{P_1 + P_2\xi^2}{P_3\xi}.$$

where $P_1 = a\gamma b^2,\ P_2 = \delta\beta,\ P_3 = b(\gamma\beta - \alpha\delta)$.

Rewrite this equation as the system

$$\eta = ctg\xi$$

$$\eta = \frac{P_1 + P_2\xi^2}{P_3\xi} = \frac{1}{P_3}\left(\frac{P_1}{\xi} + P_2\xi\right). \tag{9}$$

Then points of intersection of two curves give the eigenvalue $\lambda_n = \frac{1}{b^2}\xi^2$. The first curve is the graph of $\eta = ctg\xi$ at $\xi > 0$, and the second one is a hyperbola.

We conclude that the system (9) has infinite set of roots and these roots are real and different, i.e. ¸$\lambda_n - \lambda_m \neq 0$ if $m \neq n$ and $\lambda_n > \lambda_m$ as $n > m$. Thus, $\{\lambda_n\}$ form an increasing sequence.

These roots are $0 < \xi_1 < \frac{\pi}{2}$ and $\xi_n = \xi_1 + (n-1)\pi,\ n = 1, 2, 3, \ldots$. Then eigenvalues have the form $\lambda_n = \frac{1}{b^2}\left[\xi_1 + (n-1)\pi\right]^2$.

Corresponding eigenfunctions have the form

$$Y_n(y) = \left(\alpha \sin\sqrt{\lambda_n} y - \beta\sqrt{\lambda_n}\cos\sqrt{\lambda_n} y\right) A_n \qquad (10)$$

where A_n are constants.

Let's prove that the system of functions $\{Y_n(y)\}$ (10) of the problem (7) is orthogonal in the segment $[0,b]$.

The orthogonality of the system (10) is proven as the work in [11].

At $n=m$, without any loss of generality supposing $A_n=1$, we obtain

$$\|Y_n(y)\|^2 = \int_0^b Y_n^2(y)\,dy = \int_0^b \left(\alpha \sin\sqrt{\lambda_n} y - \beta\sqrt{\lambda_n}\cos\sqrt{\lambda_n} y\right)^2 dy$$

$$= \frac{1}{2}\left(\alpha^2 b + \beta^2\lambda_n b - \alpha\beta\right) + \frac{\beta^2\lambda_n - \alpha^2}{4\sqrt{\lambda_n}}\sin 2\sqrt{\lambda_n} b + \frac{\alpha\beta}{2}\cos 2\sqrt{\lambda_n} b.$$

The general solution of the equation (6) has the form

$$X_n(x) = C_{1n}e^{-k_n x} + e^{\frac{1}{2}k_n x}\left(C_{2n}\cos\nu_n x + C_{3n}\sin\nu_n x\right) \qquad (13)$$

where $k_n = \sqrt[3]{\lambda_n}$, $\nu_n = \frac{\sqrt{3}}{2}k_n$, $C_{in}\ (i=1,2,3)$ are arbitrary constants.

Then the function

$$U_n(x,y) = X_n(x)Y_n(y)$$

satisfies the equation and conditions (2).

By virtue of linearity and homogeneity of the equation (1), the sum of particular
Solutions

$$U(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) \qquad (14)$$

will be also the solution of (1).

The function $U(x,y)$, represented by the series (14), satisfies conditions (2) since all the members of the series satisfy them.
Satisfying the boundary conditions (3), we obtain

$$\left.\begin{aligned} U_{xx}(0,y)=\varphi_1(y)=\sum_{n=1}^{\infty}X_n''(0)Y_n(y),\\ U_x(a,y)=\varphi_2(y)=\sum_{n=1}^{\infty}X_n'(a)Y_n(y),\\ U_{xx}(a,y)=\varphi_3(y)=\sum_{n=1}^{\infty}X_n''(a)Y_n(y),\end{aligned}\right\} \qquad (15)$$

Series (15) are represented the expansion of an arbitrary function $\varphi_i(y),\ i=1,2,3$ eigenvalues of the problem (7). Members $X_n''(0),\ X_n'(a),\ X_n''(a)$ are coefficients of this expansion. If functions $\varphi_i(y)$ are integrable in the segment $[0,b]$, then the expansion (15) behaves with respect to convergence like an usual Fourier trigonometrical series [11].

For determining coefficients of (15), multiply it on $Y_m(y)$ and integrate at limits $[0,b]$, then taking into account orthogonality of the system of functions $Y_m(y)$, we obtain

$$X_m''(0)=\frac{1}{\|Y_m\|^2}\int_0^b\varphi_1(\eta)Y_m(\eta)d\eta,\ X_m'(a)=\frac{1}{\|Y_m\|^2}\int_0^b\varphi_2(\eta)Y_m(\eta)d\eta,$$

$$X_m''(a)=\frac{1}{\|Y_m\|^2}\int_0^b\varphi_3(\eta)Y_m(\eta)d\eta.$$

For convenience, introduce the notations

$$B_{in}=\frac{1}{\|Y_n\|^2}\int_0^b\varphi_i(\eta)Y_n(\eta)d\eta,\quad i=1,2,3. \qquad (16)$$

Then we obtain the system of algebraic equations for determinating coefficients $C_{in}\ (i=1,2,3)$:

$$\begin{cases} k_n^2C_{1n}-\frac{1}{2}k_n^2C_{2n}+\frac{\sqrt{3}}{2}k_n^2C_{3n}=B_{1n}\\ -k_nC_{1n}e^{-k_na}+k_ne^{\frac{1}{2}k_na}\cos\left(\nu_na+\frac{\pi}{3}\right)C_{2n}+k_ne^{\frac{1}{2}k_na}\sin\left(\nu_na+\frac{\pi}{3}\right)C_{3n}=B_{2n}\\ k_n^2e^{-k_na}C_{1n}-k_n^2e^{\frac{1}{2}k_na}\cos\left(\nu_na-\frac{\pi}{3}\right)C_{2n}-k_n^2e^{\frac{1}{2}k_na}\sin\left(\nu_na-\frac{\pi}{3}\right)C_{3n}=B_{3n}.\end{cases} \qquad (17)$$

Calculations shows that

$$\Delta=\sqrt{3}k_n^5 e^{k_n a}\left[\frac{1}{2}-e^{-\frac{3}{2}k_n a}\sin\left(\nu_n a-\frac{\pi}{6}\right)\right]\neq 0.$$

Solving the system (17), substituting values of C_{in} in (14), we obtain the solution of Problem A_α in the form

$$U(x,y)=\sum_{n=1}^{\infty}\left[B_{1n}D_{1n}(x)+B_{2n}D_{2n}(x)+B_{3n}D_{3n}(x)\right]Y_n(y) \qquad (18)$$

where

$$D_{1n}(x)=\frac{\sqrt{3}k_n^3}{\Delta}\left[\frac{1}{2}e^{k_n(a-x)}+e^{-\frac{1}{2}k_n(a-x)}\cos(\nu_n a-\nu_n x)\right],$$

$$D_{2n}(x)=\frac{k_n^4}{\Delta}\left\{-e^{\frac{1}{2}k_n(a-2x)}\sin\nu_n a-e^{-k_n\left(a-\frac{1}{2}x\right)}\left[\sin\left(\nu_n a+\frac{\pi}{3}\right)+e^{\frac{3}{2}k_n a}\sin\left(\nu_n(a-x)-\frac{\pi}{3}\right)\right]\right\},$$

$$D_{3n}(x)=\frac{k_n^3}{\Delta}\left\{-e^{\frac{1}{2}k_n(a-2x)}\cos\left(\nu_n a+\frac{\pi}{6}\right)-e^{-k_n\left(a-\frac{1}{2}x\right)}\left[\sin\left(\nu_n a+\frac{\pi}{3}\right)+e^{\frac{3}{2}k_n a}\sin\left(\nu_n(a-x)-\frac{\pi}{3}\right)\right]\right\}.$$

Let's prove the uniform convergence of the series (18) with respect to both variables.

Let (x_0,y_0) be an arbitrary point of the domain D. Then

$$U(x_0,y_0)=\sum_{n=1}^{\infty}B_{1n}D_{1n}(x_0)Y_n(y_0)+\sum_{n=1}^{\infty}B_{2n}D_{2n}(x_0)Y_n(y_0)+\sum_{n=1}^{\infty}B_{3n}D_{3n}(x_0)Y_n(y_0) \qquad (19)$$

what follows

$$|U(x_0,y_0)|\leq\sum_{n=1}^{\infty}|B_{1n}Y_n(y_0)||D_{1n}(x_0)|+\sum_{n=1}^{\infty}|B_{2n}Y_n(y_0)||D_{2n}(x_0)|+\sum_{n=1}^{\infty}|B_{3n}Y_n(y_0)||D_{3n}(x_0)|. \qquad (20)$$

Denoting

$$\vartheta_i(x_0,y_0)=\sum_{n=1}^{\infty}B_{in}D_{in}(x_0)Y_n(y_0),$$

we have

$$|\vartheta_i(x_0,y_0)|\leq\sum_{n=1}^{\infty}|B_{in}Y_n(y_0)||D_{in}(x_0)|,\quad i=1,2,3.$$

Estimate $|B_{in}Y_n(y_0)|$:

$$|B_{in}Y_n(y_0)|\leq|Y_n(y_0)||B_{in}|=|Y_n(y_0)|\frac{1}{\|Y_n\|^2}\int_0^b\varphi_i(\eta)Y_n(\eta)d\eta.$$

But

$$\left|Y_n\left(y_0\right)\right|=\left|\alpha\sin\sqrt{\lambda_n}y_0-\beta\sqrt{\lambda_n}\cos\sqrt{\lambda_n}y_0\right|\le|\alpha|+|\beta|\sqrt{\lambda_n}.$$

Then we have

$$\left|B_{in}Y_n\left(y_0\right)\right|\le\frac{\left(|\alpha|+|\beta|\sqrt{\lambda_n}\right)^2}{\left\|Y_n\right\|^2}\int_0^b\left|\varphi_i\left(\eta\right)\right|d\eta.$$

Let's prove that the expression $\frac{\left(|\alpha|+|\beta|\sqrt{\lambda_n}\right)^2}{\left\|Y_n\right\|^2}$ is bounded at $n\to\infty$:

$$\frac{\left(|\alpha|+|\beta|\sqrt{\lambda_n}\right)^2}{\left\|Y_n\right\|^2}=\frac{\alpha^2+2|\alpha\beta|\sqrt{\lambda_n}+\beta^2\lambda_n}{\left\|Y_n\right\|^2}$$

$$=\frac{\alpha^2+2|\alpha\beta|\sqrt{\lambda_n}+\beta^2\lambda_n}{\frac{1}{2}\left(\alpha^2b+\beta^2\lambda_nb-\alpha\beta\right)+\frac{\beta^2\lambda_n-\alpha^2}{4\sqrt{\lambda_n}}\sin2\sqrt{\lambda_n}b+\frac{\alpha\beta}{2}\cos2\sqrt{\lambda_n}b}$$

$$=\frac{\frac{\alpha^2}{\lambda_n}+\frac{2|\alpha\beta|}{\sqrt{\lambda_n}}+\beta^2}{\frac{1}{2\lambda_n}\left(\alpha^2b-\alpha\beta\right)+\frac{1}{2}\beta^2b+\left(\frac{\beta^2}{4\sqrt{\lambda_n}}-\frac{\alpha^2}{4\lambda_n\sqrt{\lambda_n}}\right)\sin2\sqrt{\lambda_n}b+\frac{\alpha\beta}{2\lambda_n}\cos2\sqrt{\lambda_n}b}$$

We obtain from here

$$\lim_{n\to\infty}\frac{\left(|\alpha|+|\beta|\sqrt{\lambda_n}\right)^2}{\left\|Y_n\right\|^2}=\frac{\beta^2}{\frac{1}{2}\beta^2b}=\frac{2}{b}.$$

We conclude from this that for any λ_n,

$$\left|B_{in}Y_n\left(y_0\right)\right|\le\frac{2}{b}\int_0^b\left|\varphi_i\left(\eta\right)\right|d\eta.$$

Under made suppositions concerning $\varphi_i\left(y\right)$, the following inequalities

$$\left|\varphi_i\left(y\right)\right|\le\frac{M_i}{n^2},\ i=1,3\quad\left|\varphi_2\left(y\right)\right|\le\frac{M_i}{n^3}$$

hold (see [9]). Then

$$\left|B_{in}Y_n\left(y_0\right)\right| \le \frac{2}{n^2}N,\quad i=1,3,\quad \left|B_{2n}Y_n\left(y_0\right)\right| \le \frac{2}{n^3}N$$

where $N = \max M_i,\ i=1,2,3.$

Now estimate the functions *Din*(*x*0)*:* Calculations show that we obtain the following estimations:

$$\left|D_{1n}\left(x_0\right)\right| \le \frac{1}{k_n^2\left|\overline{\Delta}\right|}\left[\frac{1}{2}e^{-k_n x_0} + e^{-\frac{1}{2}k_n\left(3a-x_0\right)}\right],$$

$$\left|D_{2n}\left(x_0\right)\right| \le \frac{1}{\sqrt{3}k_n\left|\overline{\Delta}\right|}\left[e^{-k_n\left(\frac{1}{2}a+x_0\right)} + e^{-k_n\left(2a-\frac{1}{2}x_0\right)} + e^{-\frac{1}{2}k_n\left(a-x_0\right)}\right],$$

$$\left|D_{3n}\left(x_0\right)\right| \le \frac{1}{\sqrt{3}k_n^2\left|\overline{\Delta}\right|}\left[e^{-k_n\left(\frac{1}{2}a+x_0\right)} + e^{-k_n\left(2a-\frac{1}{2}x_0\right)} + e^{-\frac{1}{2}k_n\left(a-x_0\right)}\right],$$

where

$$\overline{\Delta} = \frac{1}{2} + e^{-\frac{3}{2}k_n a}\sin\left(va - \frac{\pi}{6}\right).$$

Then

$$\left|\vartheta_1\left(x_0,y_0\right)\right| \le \sum_{n=1}^{\infty}\left|B_{1n}Y_n\left(y_0\right)\right|\left|D_{1n}\left(x_0\right)\right| \le \sum_{n=1}^{\infty}\frac{2}{n^2}N\frac{1}{k_n^2\left|\overline{\Delta}\right|}\left[\frac{1}{2}e^{-k_n x_0} + e^{-\frac{1}{2}k_n\left(3a-x_0\right)}\right]$$

$$\le C_1 N\sum_{n=1}^{\infty}\frac{\frac{1}{2}e^{-k_n x_0} + e^{-\frac{1}{2}k_n\left(a-x_0\right)}}{n^{\frac{10}{3}}}.$$

One can easily be convinced that the series $\vartheta_1\left(x_0,y_0\right)$ converges absolutely. In exactly the same way, absolute convergence of other series in (19) can be proved.

This implies that the series $U\left(x_0,y_0\right)$ converges absolutely. By virtue of arbitrariness of $\left(x_0,y_0\right)$, the series (18) converges absolutely in the domain $\overline{D}$. And what is more, derivatives with respect to both variables converge since for derivatives with respect to x, the equalities

$$D_{1n}^{(p)}(x) \le \frac{\sqrt{3}}{\Delta} k_n^{p+3} \left\{ (-1)^p \frac{1}{2} e^{k_n(a-x)} + e^{-\frac{1}{2}k_n(a-x)} \cos\left[\nu_n (a-x) - p\frac{\pi}{3} \right] \right\},$$

$$D_{2n}^{(p)}(x) \le \frac{k_n^{p+4}}{\Delta} \left\{ \begin{array}{l} (-1)^{p+1} e^{\frac{1}{2}k_n(a-2x)} \sin \nu_n a - e^{-\frac{1}{2}k_n\left(a-\frac{1}{2}x\right)} \sin\left[\left(\nu_n x + \frac{\pi}{3} \right) + p\frac{\pi}{3} \right] \\ -e^{-\frac{1}{2}k_n(a+x)} \sin\left[\left(\nu_n (a-x) - \frac{\pi}{3} \right) - p\frac{\pi}{3} \right] \end{array} \right\},$$

$$D_{3n}^{(p)}(x) \le \frac{k_n^{p+3}}{\Delta} \left\{ \begin{array}{l} (-1)^{p+1} e^{\frac{1}{2}k_n(a-x)} \cos\left(\nu_n a + \frac{\pi}{3} \right) - e^{-k_n\left(a-\frac{1}{2}x\right)} \sin\left[\left(\nu_n x + \frac{\pi}{3} \right) + p\frac{\pi}{3} \right] \\ -e^{-\frac{1}{2}k_n(a+x)} \sin\left[\left(\nu_n (a-x) + \frac{\pi}{3} \right) - p\frac{\pi}{3} \right] \end{array} \right\},$$

hold.

For the functions $D_{in}^{(p)}(x),\ i=1,2,3,$ the estimations

$$\left| D_{1n}^{(p)}(x) \right| \le \frac{k_n^{p-3}}{|\Delta|} \left[\frac{1}{2} e^{-k_n x} + e^{-\frac{1}{2}k_n(3a-x)} \right],$$

$$\left| D_{2n}^{(p)}(x) \right| \le \frac{k_n^{p-1}}{\sqrt{3}|\Delta|} \left[e^{-k_n\left(\frac{1}{2}a+x\right)} + e^{-k_n\left(2a-\frac{1}{2}x\right)} + e^{-\frac{1}{2}k_n(a-x)} \right],$$

$$\left| D_{3n}^{(p)}(x) \right| \le \frac{k_n^{p-2}}{\sqrt{3}|\Delta|} \left[e^{-k_n\left(\frac{1}{2}a+x\right)} + e^{-k_n\left(2a-\frac{1}{2}x\right)} + e^{-\frac{1}{2}k_n(a-x)} \right]$$

are valid where $0 < x < a$ and $p = 1,2,3.$

Estimate derivates with respect to x:

$$\frac{\partial^3 U}{\partial x^3} = \sum_{n=1}^{\infty} \left[B_{1n} D_{1n}'''(x) + B_{2n} D_{2n}'''(x) + B_{3n} D_{3n}'''(x) \right] Y_n(y),$$

$$\left| \frac{\partial^3 U}{\partial x^3} \right| \le \sum_{n=1}^{\infty} \left| B_{1n} Y_n(y_0) \right| \left| D_{1n}'''(x_0) \right| + \sum_{n=1}^{\infty} \left| B_{2n} Y_n(y_0) \right| \left| D_{2n}'''(x_0) \right| + \sum_{n=1}^{\infty} \left| B_{3n} Y_n(y_0) \right| \left| D_{3n}'''(x_0) \right|.$$

(21)

Then

$$\vartheta_i'''(x_0, y_0) = \sum_{n=1}^{\infty} B_{in} D_{in}'''(x_0) Y(y_0),$$

$$|\vartheta_i'''(x_0, y_0)| \le \sum_{n=1}^{\infty} |B_{in} Y(y_0)| |D_{in}(x_0)| \le \sum_{n=1}^{\infty} \frac{2}{n^2} N \frac{k_n}{|\Delta|} \left[\frac{1}{2} e^{-k_n x_0} + e^{-\frac{1}{2} k_n (3a - x_0)} \right]$$

$$= C_2 N \sum_{n=1}^{\infty} \frac{\frac{1}{2} e^{-k_n x_0} + e^{-\frac{1}{2} k_n (3a - x_0)}}{n^{\frac{4}{3}}}.$$

This series converges that's why the series $\vartheta_i'''(x_0, y_0)$ converges absolutely. By the same way one can prove absolute convergence of other series in (21). Since $\frac{\partial^3 U}{\partial x^3} = \frac{\partial^2 U}{\partial y^2}$, the absolute convergence of the second derivative with respect to y of the series (18) can beproved analogously.

In all the expressions $D_{in}^{(p)}(x)$ for $p = 3,$ the identity

$$D_{in}^{(3)}(x) + \lambda_n D_{in}(x) = 0, \quad i = 1, 2, 3$$

is valid.

For the function $D_{in}(x)$, the identity

$$\begin{bmatrix} D_{1n}''(0) & D_{1n}'(a) & D_{1n}''(a) \\ D_{2n}''(0) & D_{2n}'(a) & D_{2n}''(a) \\ D_{3n}''(0) & D_{3n}'(a) & D_{3n}''(a) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

holds which is verified immediately.

Thus, we have proved the following

Theorem 2. If $\varphi_i(y) \in C^1[0, b],\ i = 1, 3,\ \varphi_2(y) \in C^2[0, b],$ and $\varphi_j(0) = \varphi_j(b) = 0,\ j = 1, 2, 3,$ then the solution of Problem A_α exists and is represented by the series (18).

Substituting values of B_{in} from (16) in (18), we obtain the solution of Problem A_α in the form

$$U(x,y)=\int_0^b K_1(x,y,\eta)\varphi_1(\eta)d\eta+\int_0^b K_2(x,y,\eta)\varphi_2(\eta)d\eta+\int_0^b K_3(x,y,\eta)\varphi_3(\eta)d\eta$$

where

$$K_i(x,y,\eta)=\sum_{n=1}^{\infty} D_{in}\frac{1}{\|Y_n\|^2}Y_n(\eta)Y_n(y).$$

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