Water Waves Moving Over a Parabolic Beach

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ABSTRACT

Self similar transformations are introduced for the shallow waves moving over a parabolic beach. General properties of the motion are investigated by the phase plane analysis of the self similar equations. The nature of the singularities and the behavior of the integral curves are determined. Possible motions which are continuous or those which include a bore in their development are indicated.

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PARABOLİK BİR KUMSALDA HAREKET EDEN SU DALGALARI

ÖZET

Parabolik bir kumsalda hareket eden sığ su dalgaları için benzeşim dönüşümleri tariflenir. Faz düzleminde benzeşim denklemelerinin incelenmesi ile hareketin genel özellikleri araştırıldı. Tekil noktaların karakterleri ve integral eğrilerinin durumları hesaplandı. Sürekli olan veya girdap ihtiva eden mümkün hareketler belirtildi.

1. INTRODUCTION :

The concept of self-similarity plays a key role in investigating many of the physical phenomena. In a large number of cases, exact solutions of the governing equations are impossible to find and classical methods such as transform techniques, seperation of variables, etc. are of little value.

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Such is the case when the governing differential equations are nonlinear. However, there exist a class of solutions to the governing differential equations which are obtained by employing transformations that reduce the original partial differential equations to ordinary differential equations. Such motions are designated as self similar solutions. It is of interest to point out that this method of analysis forms the basis of blast wave theory (Sedor 1959).

Shallow water equations (Stoker 1957) governing the propagation of long waves like a Tsunami over shallow bottom have been the subjeet of analysis by many authors in past. Most of these analyses are restricted by the small amplitude assumptions and thus are not particularly suitable to deal with the waves of Tsunami type. In the present analysis, the governing partial differential equations are examined with the motivation of reducing them to ordinary differential equations, although other methods using dimensional analysis (Sedov 1959) are also possible. Section 2 deals with the appropriate transformations necessary to produce this reduction and consequent simplification of the problem. However, the resulting ordinary differential equations are nonlinear and exact methods of analysis are not available and one has to resort to numerical methods. But these equations are of the autonomous type (Hayashi 1964) and a great deal of information can be obtained about the structure of these equations without actually solcing them. This is done in section 3 and singularity of these nonlinear equations are investigated. Section 4 deals with the appropriate integral curves which describe various physical processes and solutions which include a bore in their development are indicated.

2. BASIC EQUATIONS AND SIMILARITY TRANSFORMATION :

In this article, we study the propagation of waves in shallow water. If x denotes the horizontal distance, t, the time measure, equations governing the disturbed depth $g^{-1}H$ and the fluid velocity u are given by (Stoker 1957)

$$U_{,t}+\left(H+\frac{U^2}{2}\right)_{,x}=h_{,x} \tag{2.1}$$

and

$$H_{,t}+(UH)_{,x}=0 \tag{2.2}$$

where $g^{-1}H(x)$ denotes the undisturbed depth.

In the following, we restrict our attention to beaches where the undisturbed depth is a polynominal function of the horizontal distance, i.e.,

$$h = h_0\ (1 - x/L)^{m} \tag{2.3}$$

where m is a constant.

Shallow water equations (2.1) and (2.2) admit similarity transformations and the resulting motions have the distinguishing property that there is similarity in the motion itself. For such motions, there exist universal curves of the dimensionless state variables

$$D = \frac{H}{h}; \qquad M = \frac{U}{h^{1/2}} \tag{2.4}$$

in terms of a new dimensionless co-ordinate

$$\xi = \xi\ (x,t) \tag{2.5}$$

called the simularity variable. In terms of this variable, the shapes of D and M remain unaltered for all time.

Contrary to the conventional approach (Sedov 1959) of seeking the similarity transformation through dimesional considerations, we shall introduce this idea by requiring that the similarity transformation reduces the partial differential equations (2.1) and (2.2) to ordinary differential equations in terms of ξ. As we will see below, that this requirement automatically determines the form of ξ as well.

When (2.4) and (2.5) are substituted into the governing equations (2.1) and (2.2), the resulting equations are

$$h^{1/2}\, \xi_{,t}\, M' + h_{,x} \left[\left(D + \frac{1}{2} M^2 - 1 \right) + \left(D + \frac{1}{2} M^2 \right)' h\, \xi_{,h} \right] = 0$$

and

$$h^{1/2}\, \xi_{,t}\, D' + h_{,x} \left[\frac{3}{2} MD + (MD)' h\, \xi_{,h} \right] = 0 \tag{2.7}$$

where the prime indicates diferentiation with respect to the similarity variable denoted in (2.5).

Examining (2.6) and (2.7), it is apperent that the necessary conditions to obtain ordinary differential equations governing D *and* η are

$$h^{1/2}\xi_{,t}=h_{,x} \tag{2.8}$$

and

$$h\,\xi_{,h}=F(\xi) \tag{2.9}$$

where F is some function of ξ only. Equation (2.8) can be integrated at once to yield

$$\xi=h^{-1/2}\,h_{,x}\ t+f(x) \tag{2.10}$$

where $f(x)$ is an orbitrary function of x. This function can be determined by labelling ξ to be zero at the instant of time when the pulse arrives at any station x, i.e.,

$$\xi=0 \text{ at } t=t_0 \tag{2.11}$$

where

$$t_0=\int_0^x h^{-1/2}(s)\,ds \tag{2.12}$$

Here t_0 indicates the arrival time of the pulse for any station x. Using (2.11) in (2.10) we have

$$\xi=h^{-1/2}\ h_{,x}\,(t-t_0) \tag{2.13}$$

Noting the restriction (2.3) on the beach shape, yields for ξ

$$\xi=-2\,m\,\frac{h_0^{1/2}}{L}\,\bar{h}^{\frac{m-1}{2m}}\quad(t-t_0) \tag{2.14}$$

where

$$\bar{h}=h/h_0 \tag{2.15}$$

Since $2m$ is a dimensionless number, this could be eliminated by defining

$$\bar{\xi}=-\xi/2m=\frac{h_0^{1/2}}{L}\,\bar{h}^{\frac{m-1}{2m}}\quad(t-t_0) \tag{2.16}$$

Substitution of (216) in to (2.9) yields

$$h\,\bar{\xi}_{,h}=\frac{1}{2m}\ [1+(m-1)\bar{\xi}] \tag{2.17}$$

Thus both tne restrictions (2.8) and (2.9) on the similarity variable are satisfied by the choice of $\bar{\xi}$ denoted in (2.16). In terms of this variable, equations (2.6) and (217) reduce to

$$M' - [1 + (m-1)\bar{\xi}]\left(D + \frac{1}{2}M^2\right)' = 2m\left(D + \frac{1}{2}M^2 - 1\right) \tag{2.18}$$

and

$$D' - [1 + (m-1)\bar{\xi}]\ (MD)' = 3m\ MD \tag{2.19}$$

where the primes indicate deferentiation with $\bar{\xi}$.

Equations (2.18) and (2.19) provide the governing equations for self similar motions when the beach shape is prescribed to be of the form (2.3). Further one needs the boundary and initial conditions for the state variables D and M.

In order to analyze these equations for a concrete case, we shall limit our analysis to the case of parabolic beaches for which the exponent m in (2.3) equals one. In this case, the similarity variable obtained in (2.16) reduces to

$$\xi = \delta = \frac{h_0^{1/2}}{L}(t - t_0) \tag{2.20}$$

The governing equations (2.18) and (2.19) transform to

$$M' - \left(D + \frac{1}{2}M^2\right)' = 2\left(D + \frac{1}{2}M^2 - 1\right) \tag{2.21}$$

and

$$D' - (MD)' = 3\ MD \tag{2.22}$$

where, now, primes are derivatives with respect to the similarity variable δ for the parabolic beach.

Solving in (2.21) and (2.22) for the derivatives M' and D', we obtain

$$D' = \frac{P(D,M)}{(1-M^2) - D} \tag{2.23}$$

and

$$M' = \frac{Q(D,M)}{(1-M^2)-D} \tag{2.24}$$

where

$$P(D,M) = D\left[3M(1-M)+2\left(D+\frac{1}{2}M^2-1\right)\right] \tag{2.25}$$

and

$$Q(D,M) = 3MD+2(1-M)\left(D+\frac{1}{2}M^2-1\right) \tag{2.26}$$

we shall analyze these ordinary diferential equations in the next section.

3. STRUCTURE OF THE SIMILARITY EQUATIONS :

Governing equations (2.23) and (2.24) provide two nonlinear ordinary diferential equations for the state variables D and M. Similarity variable δ does not appear explicitly in these and as such can ve eliminated by taking the rations of these two equations and we obtain

$$\frac{dD}{dM} = \frac{P(D,M)}{Q(D,M)} \tag{3.1}$$

Integrals of (3.1) provide solutions of the state variable D as a function of M. Having found these solutions, one could express these state variables as a function of δ by an additional quadrature of (2.24), i.e.,

$$\frac{d\delta}{dM} = \frac{(1-M^2)-D}{Q(D,M)} \tag{3.2}$$

Although exact solutions of (3.1) are not possible, a great deal of information about the solutions can be obtained by making a qualitative study of the integral curves of (3.1) numerically. Standard techniques of numerical investigations using isoclines (curves of constant slope) are available in the literature (Hayashi 1964).

Characteristic isoclines are given by

$$P(D,M)=0 \text{ for which } \frac{dD}{dM}=0 \tag{3.3}$$

and

$$Q(D,M)=0 \text{ for which } \frac{dD}{dM}=\infty \tag{3.4}$$

From (2.25) and (3.3) we have for the zero slope curves in the D - M plane,

$$D=0 \text{ and } D=1-\frac{3}{2}M+M^2 \tag{3.5}$$

Similarly (2.26) and (3.4) yield the following infinite slope curve

$$D=\frac{(M^3-M^2-2M+2)}{(M+2)} \tag{3.6}$$

Intermediate values of the slope provide the different isoclines of (3.1) and these illustrate the general nature of the integral curves. In particular, the points of intersection (3.5) and (3.6) are singular points of (3.1.). The ordinary differential equation (3.1), in the half plane $D \geqslant 0$, has several singular points whose locations and nature are described below.

$$D=0, \ M=1 \tag{3.7}$$

is one of the singular points. The nature of this singularity can be determined by using standard techniques (Hayashi 1964) and we find that this singularity is a saddle type singularity. Only two integral curves pass through this point, one solution being

$$D=0 \quad (3.8$$

and other entering the singularity with the slope $-2/3$. Assymptotic formulas for this integral curve can be obtained form (3.1) and (3.2) and we have

$$D=\frac{2}{3}(1-M) \tag{3.9}$$

and

$$\delta-\delta^*=\frac{-2}{3}(1-M) \tag{3.10}$$

where δ^* is an arbitrary constant.

Further, the intersection of $D=0$ curve with (3.6) yields the following singular points :

$$D=0, \quad M=\pm\sqrt{2} \tag{3.11}$$

Both of these singular points have the structure of nodes. In the neighborhood of $D=0$, $M=\sqrt{2}$, all the integral curves approach the singularity with zero slope. These curves are represented by

$$D=D_1^{*}\ (M-\sqrt{2}\,)^{3/2} \tag{3.12}$$

and

$$M-\sqrt{2}=M_1^{*}\ \exp\left[-2\sqrt{2}\,(\sqrt{2}-1)\ (3-2\sqrt{2}\,)^{-1}\ \delta\right] \tag{3.13}$$

where D^{*} and M^{*} are arbitrary constants. This point is approached as $\delta \rightarrow \infty$. This corresponds to, the time tending to infinity at and fixed station. Further, there is an exceptional integral curve which enters the point with the slope $-(\sqrt{2}-1)\ (\sqrt{2}+1)^{-1}$. This is described by

$$D=-(\sqrt{2}-1)\,(\sqrt{2}+1)^{-1}(M-\sqrt{2}) \tag{3.14}$$

and

$$M=M_2^{*}\ \exp\left[-(6-3\ \sqrt{2})\ (3-2\ \sqrt{2})^{-1}\ \delta\right] \tag{3.15}$$

Similar assymptotic relations can be retained for the other singularity at $D=0$, $M=-\sqrt{2}$. For the curves approaching the point with zero slope, we have

$$D=D_2^{*}\ (M+\sqrt{2})^{3/2} \tag{3.16}$$

and

$$M+\sqrt{2}=M_3^{*}\ \exp\left[-2\sqrt{2}\ (\sqrt{2}+1)\ \ (3+2\sqrt{2})^{-1}\ \delta\right] \tag{3.17}$$

where D_2^{*} and M_3^{*} are arbitrary. For the exceptional integral curve,

$$D=-(\sqrt{2}+1)\ (\sqrt{2}-1)^{-1}\ (M+\sqrt{2}) \tag{3.18}$$

and

$$M+\sqrt{2}=M^{*}+\exp\left[-\ (6+3\ \sqrt{2})(3+2\sqrt{2})^{-1}\ \delta\ \right] \tag{3.19}$$

Again as this point is approached $\delta \rightarrow \infty$

$$D=1, \quad M=0 \tag{3.20}$$

is also a singular point. But this singular point is not a simple singularity. At this point, a saddle and a node coalesces to form a complex singularity. We shall beriefly sketch the analysis for this singularity.

If we make the substitation

$$D=1+\bar{d}, \quad M=m \tag{3.21}$$

in (3.1), we have from (3.1), (2.25) and (2.26),

$$\frac{d\bar{d}}{dm}=\frac{P(\bar{d},m)}{q(\bar{d},m)} \tag{3.22}$$

where

$$P(\bar{d},m)=2\bar{d}+3m-2m^2+2\bar{d}^2+3m\bar{d}-2m^2\bar{d} \tag{3.23}$$

and

$$q(\bar{d},m)=2\bar{d}+3m+m^2+m\bar{d}-m^3. \tag{3.24}$$

In (3.23) and (3.24), linear terms $2\bar{d}+3m$ being the same produces the complexity at the singularity. All curves enter this singularity with the slope $-3/2$. These curves can be analyzed by using the substitution,

$$\bar{d}=-\frac{3}{2}m+Am^2+\phi m \tag{3.25}$$

where the constant A and the function $\phi\ (m)$ are to be determined. Substituting (3.25) into (3.22) and retaining only the first order terms, we have,

$$m^2\frac{d\phi}{dm}\cong\frac{20A-11}{8A-1}+\frac{20}{(8A-2)}\phi+\frac{(6-2A-16A^2)}{8A-2}m+0(m^2,\phi^2,\phi m) \tag{3.26}$$

The constant term can be eliminated by the choice

$$A=\frac{11}{20} \tag{3.27}$$

Then (3.26) reduces to

$$m^2\frac{d\phi}{dm}=\frac{25}{3}\phi+\frac{1}{40}m \tag{3.28}$$

Singularities such as the one represented in (3.28) have been analyzed in the literature (Hayashi 1964) and they have the structure of a coalesced saddle and node.

In order to obtain the representation for tce curves approaching this singularity, a further transformation of (3.28) is required. By using the transformation

$$P^2=\frac{3}{25}m\,; \quad q=\frac{25}{3}\phi+\frac{1}{40}m \tag{3.29}$$

equation (3.28) can further be simplified and we have

$$P^2\frac{dq}{dp}=q \tag{3.30}$$

This can be integrated to yield

$$q=c\,e^{-1}/P \tag{3.31}$$

where c is arbitrary constant. Noting (3.20), (3.25), (3.27) and (3.29), equation (3.31) can be transformed back to the original variables, i.e.,

$$D-1=-\frac{3}{2}M+\frac{11}{20}M^2-\frac{3}{1000}M^3+c'M^2e^{\frac{-25}{3M}} \tag{3.32}$$

where c' is an arbitrary constant. Integral curves in this neighborhood can be sketched by choosing an initial D_0 and M_0 and hence the constant c' in (3.22). The variation of the similarity parameter δ can be computed with the aid of (3.2) and (3.32). By using the same linearization procedure indicated in (3.21), we have

$$M=M_5\,e^{\frac{-6}{5}\delta} \tag{3.33}$$

where $M_5{}^*$ is an arbitrary constant. Again this point is approached with infinite value of δ

Other singularities are located at infinitely distant points. A saddle type singularity exists at

$$D=\infty\,; \quad M=-2 \tag{3.34}$$

The two solutions that pass through this singularity are given by

$$D = \infty \tag{3.35}$$

and

$$D = -\frac{2}{(M+2)} \tag{3.36}$$

For the latter, we have

$$M + 2 = M_6{}^* \; e^{-6\bar{\delta}} \tag{3.37}$$

The singularity at

$$D = 0, \quad M = +\infty \tag{3.38}$$

is a saddle with the two solutions that pass through this point being

$$D = 0 \text{ and } M = \infty \tag{3.39}$$

By choosing

$$D = \frac{1}{y}; \quad M = \frac{1}{x} \tag{3.40}$$

equation (3.1) can be transformed into a form where in the infinitely distant singular points of (3.1) are brought to the origin. We find that the origin in the new co-ordinates is a complex singularity of the same structure as that of the point $D = 1$, $M = 0$. Integral curves approach the point

$$D = \infty \quad ; \quad M = -\infty \tag{3.41}$$

with a nodal structure, while they have a saddle type distribution near

$$D = \infty \quad ; \quad M = +\infty \tag{3.42}$$

Near the point indicated in (3.41), they have the asymptotic representation

$$D = cM^2 \tag{3.43}$$

with c being a constant.

Having thus computed the locations and nature of the singularities of (3.1), it remains only to indicate the course of the integral curves away from the singularities. This is taken up in the next section.

4. DISCUSSION OF THE INTEGRAL CURVES :

Figure 2 displays the integral curves in the $D-M$ plane obtained by numerical integration of (3.1). A fourth order Runge-Kutta method (Carnahan, Luher, Wilkes 1969) is employed in all the numerical computati-

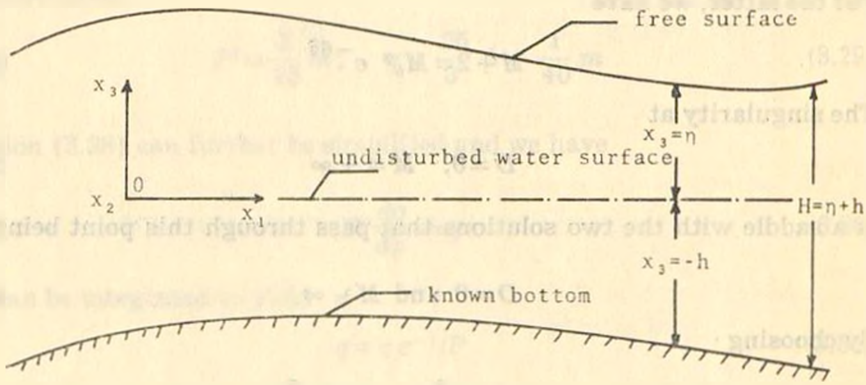
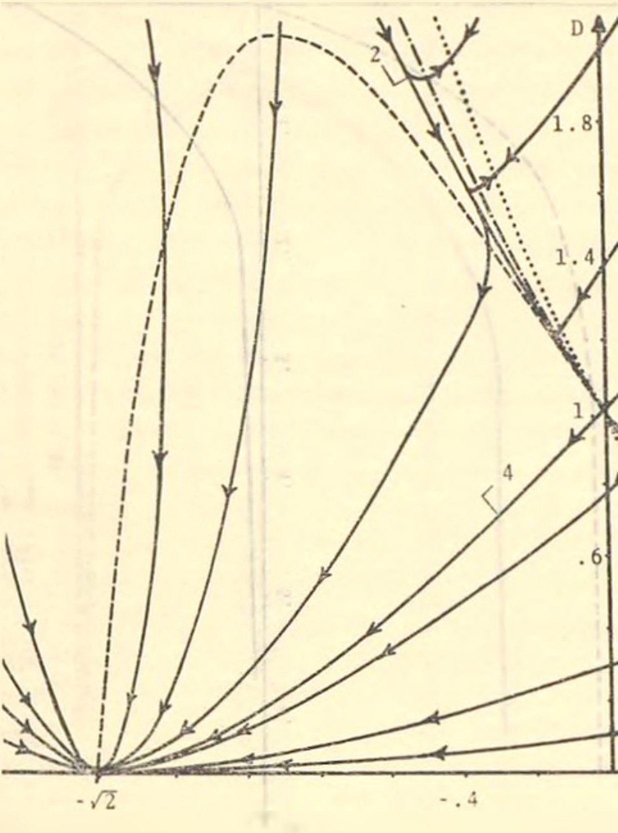


Figure 1.

ons. Integral curve (marked 1 in figure 2) originates at the saddle at $D=0$, $M=1$, with the slope $-2/3$ and enters the saddle-node at $D=1$, $M=0$ with the slope $-3/2$. For $M<0$, the same curve marked 2 enters the node at $D=\infty$, $M=-\infty$. Curve markad 3 strats at $D=1$, $M=0$ with unit slope and enters the node at $D=0$, $M=+\sqrt{2}$ with zero slope. Similarly for $M<0$, solution curve 4 starts at $D=1$, $M=0$ with unit slope and eters the node at $D=0$, $M=-\sqrt{2}$. Solution curve 5 which has the assymtotic representations (3.14) and (3.15) asymtotes to the curve 1. Similarly, the exceptional curve 6 starting at the other node located at $M=-\sqrt{2}$, $D=0$, asymtotes to the curve 2. These six curves divide the $D-M$ plane into several regions in each of which the integral curves have markedly different behaviours. All curves originating to the left of the solution curve 5, at the node at $M=\sqrt{2}$, $D=0$ joins the curve 1. All curves originating to the rigth of this curve but to the left of curve 3, attain a maximum and joins the curve 1. All curves to the right of the curve numbered 3 asymptotically joins the curve numbered 2 which enter$D=\infty$, $M=-\infty$.

For$M<0$, curves starting form $M=-\sqrt{2}$, $D=0$ to the right of the solution curve numbered 4 enters the singularity at $D=1$, $M=0$. Rest of them approach $D=\infty$ and $M=-\infty$.


D
2
1.8
1.4
1
4
.6
$-\sqrt{2}$
-.4

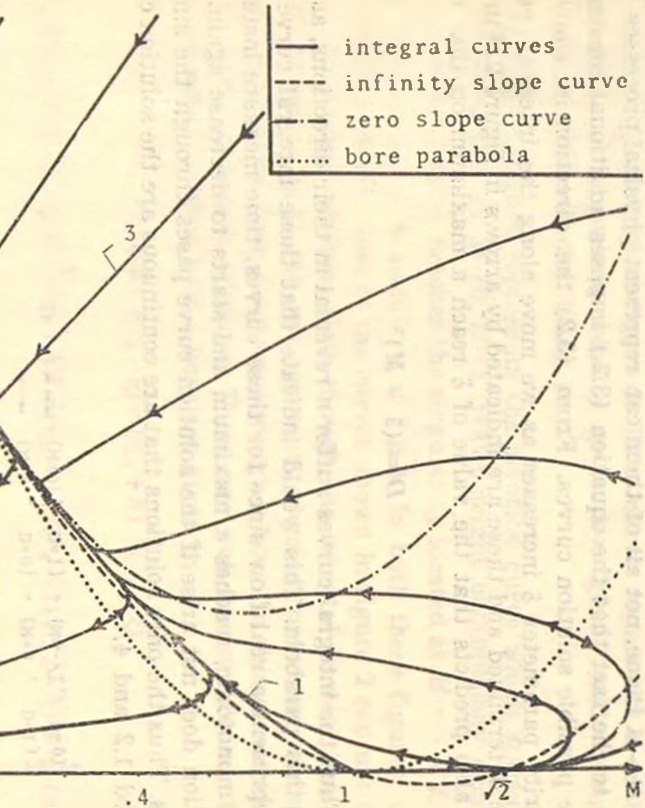

integral curves
infinity slope curve
zero slope curve
bore parabola
3
1
.4
1
√2
M

Figure 2.

Although each one of the integral curves represents a possible solution in the D—M Plane, not all of them can represent physical processes· This is due to the fact that the equation (3.2.) imposes additional constraints on the possible solution curves. From (3.2) the direction in which the similarity parameter δ increases as we move along the integral curves can be determined and these are indicated by arrows in figure 2. Further, (3.2) also predicts that the value of δ reach a maximum on the curve

$$D = (1 - M)^2 \qquad (4.1)$$

Thus the integral curves suffer a reversal in their directions, as they croos this parabola. This would indicate that these integral curves can not represent a real flow since for these curves, time measure instead of being monotonic, reaches a maximum and starts to decrease again. This situation does not arise if the solution curve pases through the singular points. Thus the only solutions that are continuous are the solution curves marked 1,2 and 4.

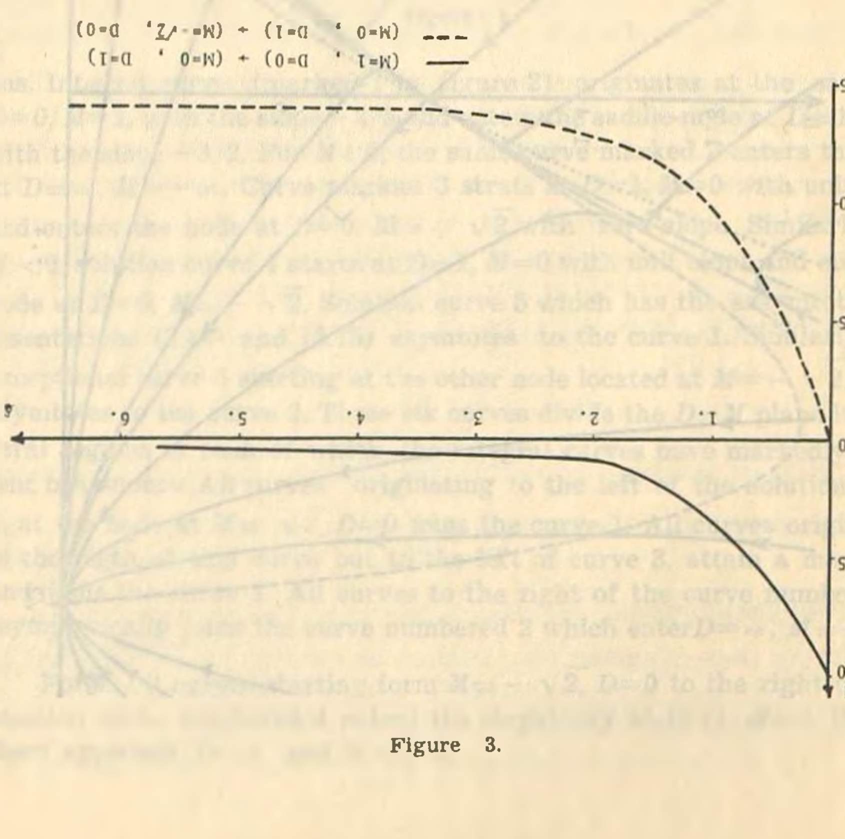


Figure 3.

Solution curve starting at $M=1$, $D=0$ and ending at the singularity at $M=0$, $D=1$ is plotted in figure 3. This curve represents a continous solution with initial positive velocity given by $(gh)^{1/2}$ and ending in a state of rest with velocity being zero.

Similarity figure 4 represents the solution curve which starts with zero initia velocity at $M=0$, $D=1$. Wave height continuously decreases along the curve until it reaches the singularity located at $M=-\sqrt{2}$, $D=0$ which represnt $\eta=-h$ and $u=-\sqrt{2gh}$. In both these figures, waves are of infinite extent. Rest of the curves shown in figure 2 can not represents a continuous solution. Even though the integral curves that intersect the parabola (4.1) can not represent physical processes, it may be possible to continue these solutions through a bore. In this connection we need the bore transition relations. These are well known (Stoker 1957). The jump relations that hold a bore are given by

$$(\eta_B+h)(U_B-\dot{\xi})=(\eta_A+h)(U_A-\dot{\xi}) \tag{4.42}$$

and

$$(\eta_B+h)(U_B-1)(U_B-\dot{\xi})+\frac{1}{2}(\eta_B+h)^2=(\eta_A+h)(U_A-1)(U_A-\dot{\xi})+\frac{1}{2}(\eta+h)^2$$

(4 3)

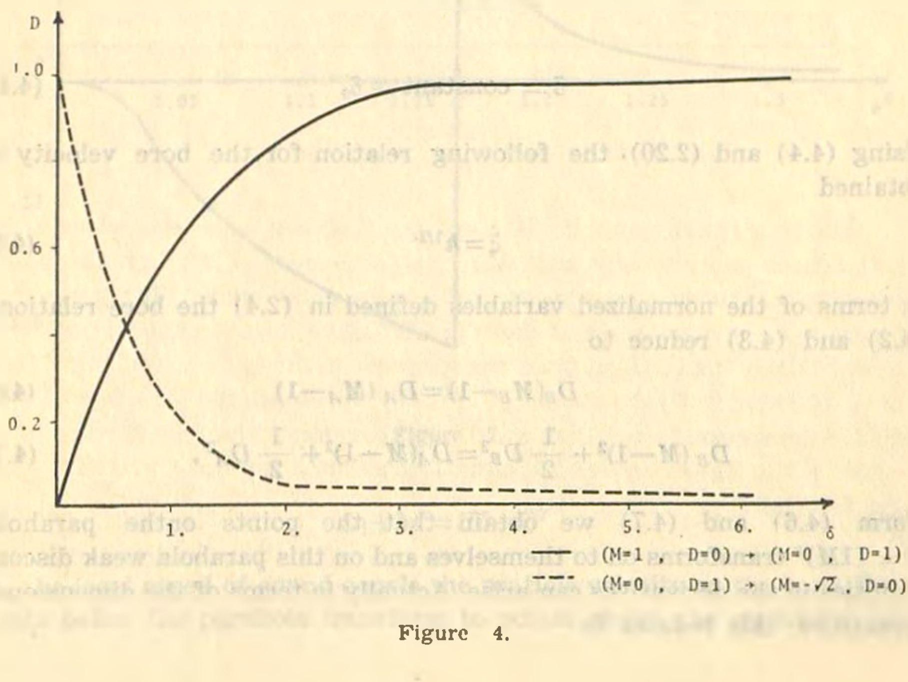


Figure 4.

where the quantities with the subscripts A and B represent the physical variables ahead and behind of the bore respectively (Figure 5). Above, ξ represents the bore velocity. The location of the bore is determined by

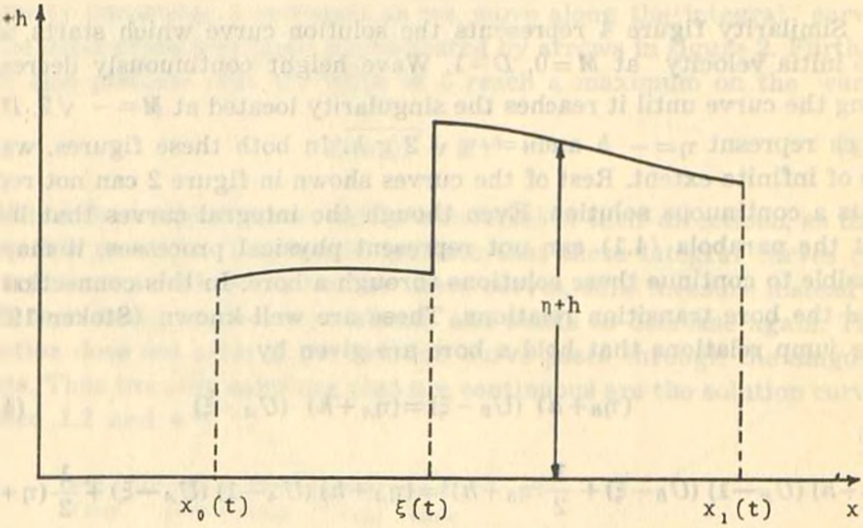


Figure 5.

$$\delta = \text{constant} = \delta_0 \tag{4.4}$$

Using (4.4) and (2.20), the following relation for the bore velocity is obtained

$$\dot{\xi} = h^{1/2} \tag{4 5}$$

In terms of the normalized variables defined in (2.4) the bore relations (4.2) and (4.3) reduce to

$$D_B(M_B-1) = D_A\,(M_A-1) \tag{4.6}$$

$$D_B\,(M-1)^2 + \frac{1}{2}\,D_B{}^2 = D_A(M-1)^2 + \frac{1}{2}\,D_A{}^2\,. \tag{4.7}$$

Form (4.6) and (4.7) we obtain that the points onthe parahola $D = (1M)^2$ transforms on to themselves and on this parabola weak discontinuities in the derivatives can arise. Actually in terms of the dimensional quantities, this reduces to

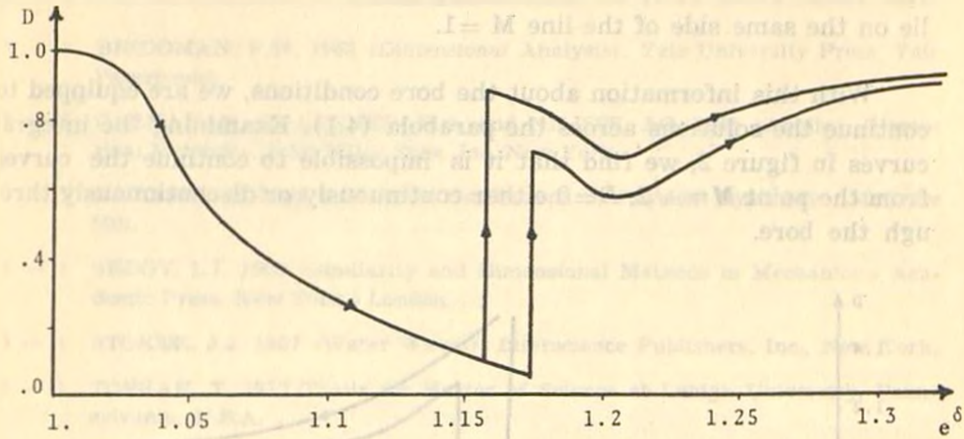


Figure 6.

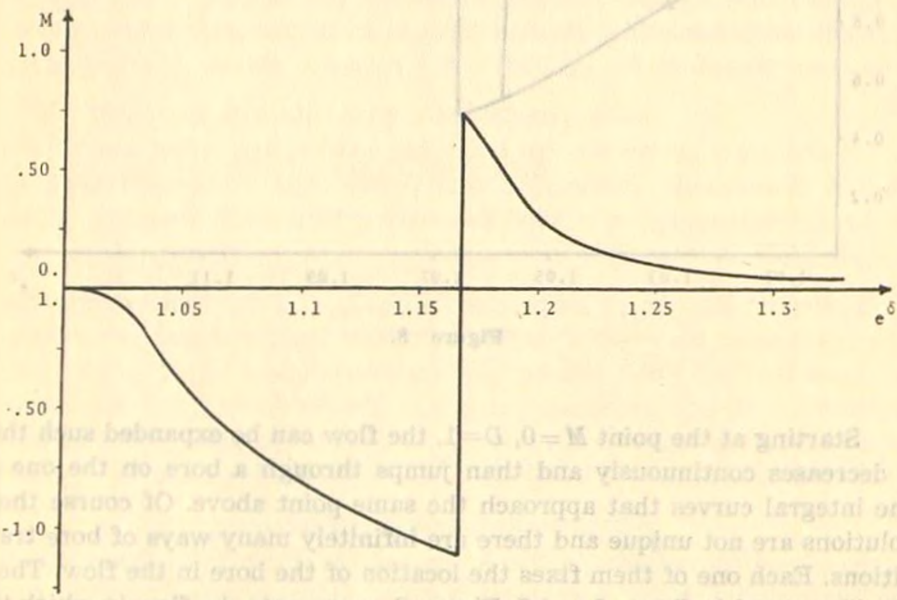


Figure 7.

$$(\eta + h)^2 = (U - \xi)^2 \tag{4.8}$$

i.e., the local speed of sound equals the praticle velocity on the bore. This points below the parabola transform to points above the parabola and vice versa. Form (4.6) we have that physical varaiables across the bore lie on the same side of the line M $=1$.

With this information about the bore conditions, we are equipped to continue the solutions across the parabola (4.1). Examining the integral curves in figure 2, we find that it is impossible to continue the curves from the point $M=\sqrt{2}$, $D=0$ either continuously or discontinuously through the bore.

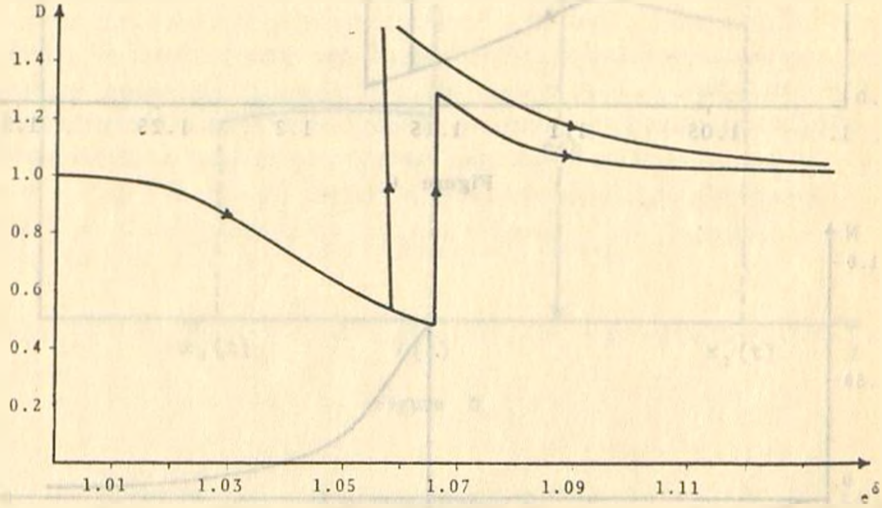


Figure 8.

Starting at the point $M=0$, $D=1$, the flow can be expanded such that η decreases continuously and than jumps through a bore on the one of the integral curves that approach the same point above. Of course these solutions are not unique and there are infinitely many ways of bore transitions. Each one of them fixes the location of the bore in the flow. These are illustrated in fiyure 6 and 7. Figure 6, represents the flow in which the flow is compressed such that wave height is below the free surface of the water, while figure 7 gives the situation in which the hawe height is above the free surface of the water.

The above curves illustrate some of the possible similarity solutions either with or without the bore transitions.

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