A Circular Beam on a Wieghardt Type Elastic Foundation

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Starting from the two-dimensional formulation of the Wieghardt type elastic foundation, the deflection of the foundation subjected to an arbitrarily distributed circular load is obtained hy the method of harmonic anaiysis. Based on this study, the solution is given in the form of the Fourier series for the problem of circular beam resting on a Wieghardt type elastic foundation and subjected to an arbitrarily distributed load. Numerical results are presented for the two cases of contcentrated loadings.

1. NOTATIONS

EI	bending rigidity
GJ	torsional rigidity
I_0, I_n	modified Bessel functions of the first kind
K_0, K_n	modified Bessel functions of the second kind
M_l	concentrated moment
M	bending moment
M_0, M_{ns}, M_{nc}	Fourier coefficients of
Q_l	concentrated load
Q	shearing force
Q_0, Q_{ns}, Q_{nc}	Fourier coefficients of
T	torsional moment
T_0, T_{ns}, T_{nc}	Fourier coefficiets of
a	radius of the circular loading and of the circular beam

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k spring constant of the foundation
p foundation pressure
p_0, p_{ns}, p_{nc} Fourier coefficients of
p nondimensional pressure of the foundation
q arbitrarily distributed circular load
q_0, q_{ns}, q_{nc} Fourier coefficiets of
r radial coordinate
t tension of the surface
w nondimensional deflection of the foundation
w deflection of the foundation
w_0, w_{ns}, w_{nc} Fourier coeficients of
$\bar{w}$ deflection of the circular beam
$\bar{w}_0, \bar{w}_{ns}, \bar{w}_{nc}$ Fourier coefficients of
w_i deflection of the foundation inside of the circular loading
w_{i0}, w_{ins}, w_{inc} Fourier coefficients of w_i
w_d deflection of the foundation outside of the circular loading
w_{d0}, w_{dns}, w_{dnc} Fourier coeficients of w_d
α, β nondimensional parameters of the foundation and the beam
β_t angle of twist
$\bar{\beta}$ nondimensional angle of twist
$\bar{\beta}_0, \bar{\beta}_{ns}, \bar{\beta}_{nc}$ Fourier coefficients of β
ν ratio of torsional rigidity to bending rigidity

2. INTRODUCTION

The Winkler hypotesis assumes that an elastic foundation consists of unconnected elastic springs and that the foundation pressure is proportional to the deflection of the foundation. This elementary theory has been the subject of some criticism because of discontinuities in the deflactions of the foundation surface at the boundaries of a finite structure. A more rational hypothesis was suggested by Wieghardt [1] using two parameters. On the basis of this hypothesis, the deflection of the foundation surface w and the foundation pressure p are related according to,

$$p = kw - t\Delta w,$$

where the constants k and t represent the properties of the elastic foundation. This equation includes the Winkler hypothesis in the special case of $t=0$. Schiel [2] pointed out that a mechanical model of this hypothesis is a liquid with a certain surface tension. Another model of the Wieghardt foundation given by Loof [3] consists of springs coupled to one another with elements which transmit a shear force proportional to the difference between the deflections of two consecutive elements. This type of elastic foundation can also correspond to the system of springs with a spring constant k and a membrane with a tension t layed on them [4].

Solutions for the beam resting on a Wieghardt type elastic foundation was obtained by Ylinen and Mikkola [5] considering a beam of finite length and taking the effect of the shear stresses on the curvature of the beam. On the other hand Smith [6] obtained the static buckling load for a beam pinned at both ends and resting on this type of foundation. Study of the influence of a Wieghardt type elastic foundation on the stability of cantilever and clamped - hinged beams subjected to either a uniformly or a linearly distributed tangential forces was made by Anderson [7]. In a recent investigation the behavior of the foundation under a semi-infinitely long beam subjected to three cases of loading has been obtained by the author [4]. All these investigations have been carried out considering the problem as one dimensional.

Solutions for the surface of the Wieghardt foundation subjected to a loading distributed on a circular area were given by Loof [3] using the equation

$$\Delta w - s^2 w = -\frac{s^2}{k} p, \tag{1}$$

where $s^2=k/t$, and $2/s$ is called cooperating width of coupled springs as a comparing value for the maximum deflections of Winkler and Wieghardt types of elastic foundations subjected to a line load. No further two dimensional solution is avaible so far the author knows.

As it will be dealt below, it is interesting that, although the Wieghardt type elastic foundation does not permit any discontinuity in the deflection of foundation surface, concentrated foundation pressures appear at the discontinuities of the slope of the deflection, such as at the boundaries of the structures, because of the surface tension t.

In the present paper, a solution is given for a circular beam resting on a Wieghardt type elastic foundation using the Fourier series. For the Winkler type, the problem was solved by Volterra [8] and by Bechert [9] using the same method.

3. ANALYSIS

3.1. Circular line load

We consider an arbitrarily distributed circular line load on a circle of radius a in the form a harmonic series,

$$p(\theta)=p_0+\sum p_{ns}\sin n\theta+\sum p_{nc}\cos n\theta, \tag{2}$$

on a Wieghardt foundation. The summation will be carried out with respect to $n=1,2,\ldots$. Assuming the deflection of the foundation in a similar form, i.e.,

$$w(r,\theta)=w_0(r)+\sum w_{ns}(r)\cdot\sin n\theta+\sum w_{nc}(r)\cos n\theta, \tag{3}$$

and substituting (2) and (3) into (1), we obtain the following equations for the unknown functions w_0, w_{ns} and w_{nc};

$$w''_0+\frac{w_0'}{r}-s^2w_0=0,$$

$$w''_{ns}+\frac{w'_{ns}}{r}-\left(s^2+\frac{n^2}{r^2}\right)w_{ns}=0,$$

$$w''_{nc}+\frac{w'_{nc}}{r}-\left(s^2+\frac{n^2}{r^2}\right)w_{nc}=0, \tag{4}$$

where the prime denotes derivatives with respect to r. The solutions of (4) are the modified Bessel functions [10]. Remembering that the deflaction of the middle point of the circular load has to be finite and the deflection has to diminish as r increases, we obtain the solutions in the following from,

$$w_i=w(r,\theta)=A_0I_0(sr)+\sum A_{ns}I_n(sr)\sin n\theta+\sum A_{nc}I_n(sr)\cdot\cos n\theta \qquad \text{for } r\le a,$$

$$w_d=w(r,\theta)=B_0K_0(sr)+\sum B_{ns}K_n(sr)\cdot\sin n\theta+\sum B_{nc}K_n(sr)\cdot\cos n\theta \qquad \text{for } r\ge a,$$

where A and B represent the constants of integration. The boundary conditions comprise, the continuity of the deflection under the load, i.e.,

$$w_i=w_d \text{ for } a=a, \tag{5}$$

and the discontinuity at the slop of the deflection which can be obtain by integration of (1) between $r=a-0$ and $r=a+0$ or by writing the equilibrium at $r=a$, which yields,

$$\frac{\partial w_d}{\partial r}-\frac{\partial w_i}{\partial r}=-\frac{s^2}{k}p \qquad \text{for } r=a. \tag{6}$$

Furthermore, the following relation has been used for reducing and rearranging the boundary conditions (5) and (6),

$$I_{n+1}(sr)\cdot K_n(sr)+I_n(sr)\cdot K_{n+1}(sr)=\frac{1}{sr}\ .$$

Consequently we obtain

$$w_i=\frac{s^2a}{k}\Big[p_0\,k\,(as)\quad I_0(rs)+\sum p_{ns}\,K_n(as)\cdot I_n(rs)\ \sin n\,\theta+$$

$$+\sum p_{nc}\,K_n(as)\cdot I_n(rs)\cdot\cos n\,\theta\Big],$$

$$w_d=\frac{s^2a}{k}\Big[p_o\,I_o(as)\cdot K_o(rs)+\sum p_{ns}\,I_n(as)\cdot K_n(rs)\cdot\sin n\theta+$$

$$+\sum p_{nc}\,I_n(as)\cdot K_n(rs)\cdot\cos n\theta\Big]\ . \tag{7}$$

The deflection of the circle of radius a as derived from (7) is

$$\bar{w}(\theta)=w_i(a,\theta)=w_d(a,\theta)=$$

$$=\bar{w}_o+\sum\bar{w}_{ns}\sin n\theta+\sum\bar{w}_{nc}\cos n\theta, \tag{8}$$

where

$$\bar{w}_0=\frac{s^2a}{k}\ p_o\,I_o(as)\cdot K_o(as),$$

$$\bar{w}_{ns}=\frac{s^2a}{k}\ p_{ns}\,I_n(as)\cdot K_n(as),$$

$$\bar{w}_{nc}=\frac{s^2a}{k}\ p_{nc}\,I_n(as)\ K_n(as)\cdot \tag{9}$$

3.2. Circular beam

We now consider a circular beam of a radius a subjected to an arbitrarily distributed circular load in the form of

$$q(\theta) = q_0 + \sum q_{ns} \sin n\theta + \sum q_{nc} \cos n\theta. \tag{10}$$

The equations of equilibrium at a beam element shown in Fig. 1 are,

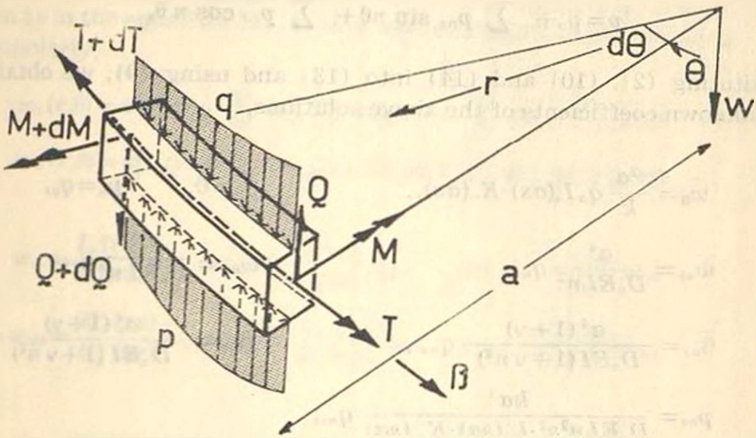

Fig. 1. Geometry and coordinate system.

$$\begin{aligned} &Q' + a(q - p) = 0, \\ &M' - T - aQ = 0, \\ &T' + M = 0, \end{aligned} \tag{11}$$

where Q, M, T and p are the shearing force, the bending moment, the torsional moment and the foundation pressure, respectively, and the prime denotes derivative with respect to θ. The relations of deformation are,

$$M = -\frac{EI}{a^2}(\beta_t + \overline{w''}), \qquad T = -\frac{GJ}{a^2}(\overline{w'} - \beta_t'), \tag{12}$$

where EI and GJ denotes the felxural and torsional rigidity of the cross section, and w and β_t are the deflection of the beam and the angle of twist of the cross section, the positive direction of which are shown in Fig. 1. From (11) and (12) we obtain two equations for $\bar{w}$ and $\beta = \beta_t a$,

$$\begin{aligned} &\overline{w^{4}} + (1+\nu)\bar{\beta}'' - \nu \overline{w''} + \frac{a^4}{EI}(p - q) = 0, \\ &\beta'' - \frac{1}{\nu}\beta - \frac{1+\nu}{\nu}\overline{w''} = 0, \end{aligned} \tag{13}$$

where $\nu = GJ/EI$. The solutions of (13) can be expressed as

$$\bar{w}=\bar{w}_0+\sum \bar{w}_{ns}\sin n\theta+\sum \bar{w}_{nc}\cos n\theta,$$

$$\bar{\beta}=\bar{\beta}_0+\sum \bar{\beta}_{ns}\sin n\theta+\sum \bar{\beta}_{nc}\cos n\theta,$$

$$p=p_0+\sum p_{ns}\sin n\theta+\sum p_{nc}\cos n\theta. \tag{14}$$

Substituting (2), (10) and (14) into (13) and using (9), we obtain for the unknown coefficients of the above solutions,

$$\bar{w}_0=\frac{s^2a}{k}q_o I_o(as)K_o(as), \qquad \bar{\beta}_o=0, \qquad p_o=q_o,$$

$$\bar{w}_{ns}=\frac{a^4}{D_nEI\,n^2}q_{ns}, \qquad \bar{w}_{nc}=\frac{a^4}{D_nEI\,n^2}q_{nc},$$

$$\bar{\beta}_{ns}=\frac{a^4(1+\nu)}{D_nEI(1+\nu n^2)}q_{ns}, \qquad \bar{\beta}_{nc}=\frac{a^4(1+\nu)}{D_nEI(1+\nu n^2)}q_{nc},$$

$$p_{ns}=\frac{ka^3}{D_nEI\,n^2s^2\cdot I_n(sa)\cdot K_n(sa)}q_{ns},$$

$$p_{nc}=\frac{ka^3}{D_nEI\,n^2s^2\cdot I_n(sa)\cdot K_n(sa)}q_{nc}, \tag{15}$$

where the abbreviation D_n denotes,

$$D_n=n^2+\nu+\frac{ka^3}{EIn^2s^2\cdot I_n(sa)\cdot K_n(sa)}-\frac{n^2(1+\nu)^2}{1+\nu n^2}.$$

The bending and torsional moments and the shearing force can be obtained also in the form of the harmonic series, i.e.,

$$M=\sum M_{ns}\sin n\theta+\sum M_{nc}\cos n\theta,$$

$$T=\sum T_{ns}\sin n\theta+\sum T_{nc}\cos n\theta,$$

$$Q=\sum Q_{ns}\sin n\theta+\sum Q_{nc}\cos n\theta, \tag{16}$$

where

$$M_{ns}=\frac{a^2\nu(n^2-1)}{D_n(1+\nu n^2)}q_{ns}, \qquad M_{nc}=\frac{a^2\nu(n^2-1)}{D_n(1+\nu n^2)}q_{nc},$$

$$T_{ns}=-\frac{a^2\nu(n^2-1)}{D_nn(1+\nu n^2)}q_{nc}, \qquad T_{nc}=\frac{a^2\nu(n^2-1)}{D_nn(1+\nu n^2)}q_{ns},$$

$$Q_{ns} = -\frac{a^3\nu(n^2-1)^2}{D_n n(1+\nu n^2)} q_{nc}, \qquad Q_{nc} = \frac{a^3\nu(n^2-1)^2}{D_n n(1+\nu n^2)} q_{ns}.$$

The deflection of the surface of the foundation under the circular beam is given as in the equation (7), or it can be expressed in the following form similarly,

$$w_i(r,\theta) = w_{io}(r) + \sum w_{ins}(r)\cdot\sin n\theta + \sum w_{inc}(r)\cdot\cos n\theta,$$

$$w_d(r,\theta) = w_{do}(r) + \sum w_{dns}(r)\cdot\sin n\theta + \sum w_{dnc}(r)\cdot\cos n\theta, \tag{17}$$

where

$$w_{io} = \frac{I_o(rs)}{I_o(as)}\bar{w}_o, \quad w_{ins} = \frac{I_n(rs)}{I_n(as)}\bar{w}_{ns}, \quad w_{inc} = \frac{I_n(rs)}{I_n(as)}\bar{w}_{nc},$$

$$w_{do} = \frac{K_o(rs)}{K_o(ar)}\bar{w}_o, \quad w_{dns} = \frac{K_n(rs)}{K_n(as)}\bar{w}_{ns}, \quad w_{dnc} = \frac{K_n(rs)}{K_n(as)}\bar{w}_{nc}.$$

4. NUMERICAL EXAMPLES AND DISCUSSION

The numerical computation was carried out on the B3700 Computer at the Computer Center of Technical University of Istanbul, and for the purpose of numerical application two special cases of loadings are chosen, i.e., a concentrated load Q_l acting at $\theta=0$ (Q_l–loading), for which

$$q_0 = \frac{Q_l}{2\pi a}, \quad q_{ns} = 0, \quad q_{nc} = \frac{Q_l}{\pi a},$$

and a concentrated moment M_l directed outwards at $\theta=0$ (M_l–loading), for which

$$q_o = 0, \quad q_{nc} = 0, \quad q_{ns} = \frac{M_l n}{\pi a^2}.$$

Further, the numerical value of ν is assumed as 0.769, which corresponds to the circular cross section with a Poisson's ratio of 0.3. The remaining parameters of the foundation and the beam can be expressed in two non-dimensional parameters, namely,

$$\alpha = \frac{ka^3}{EIs^2}, \quad \beta = sa,$$

for which have been given various numerical values in the computation.

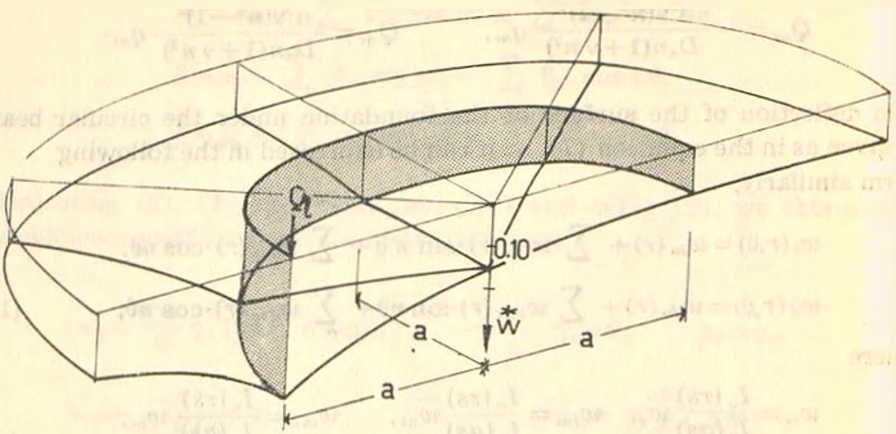

Fig. 2 (a). The surface of the foundation (Q_l—loading- for $\alpha=1.0$ and $\beta=1.0$.

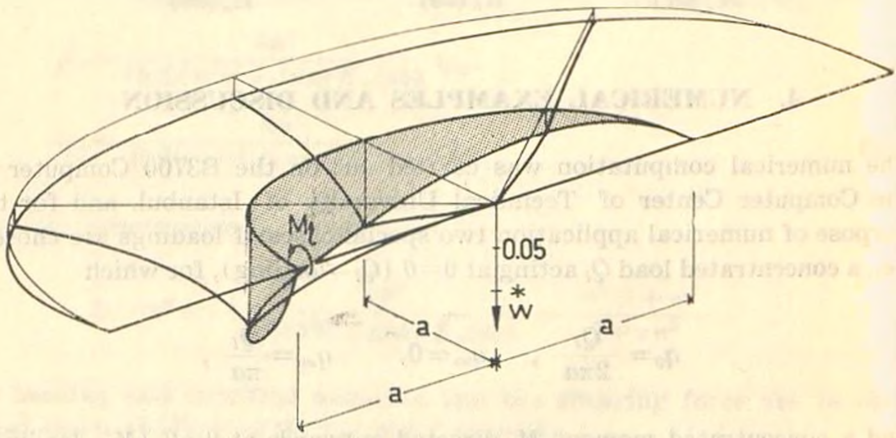

Fig. 2 (b). The surface of the foundation (M_l—loading) for $\alpha=1.0$ and $\beta=1.0$.

Fig. 2 shows the shape of the foundation surface as well as the deflection of the beam. Because of symmetry (Q_l—loading) or antisymmetry (M_l—loading) the half of the surface is drawn. The advantages of the Wieghardt type of foundation model can be seen clearly in Fig. 2. The surface of the foundation has no discontinuity, but only its slope becomes discontinuous on the contacting curve of the foundation and the beam, where the foundation pressure comes into being. However, this discontinuity appears here because the beam touches the foundation along a circular line, and it will vanish if the touching takes place on a contacting surface. The deflection of the foundation in the radial direc-

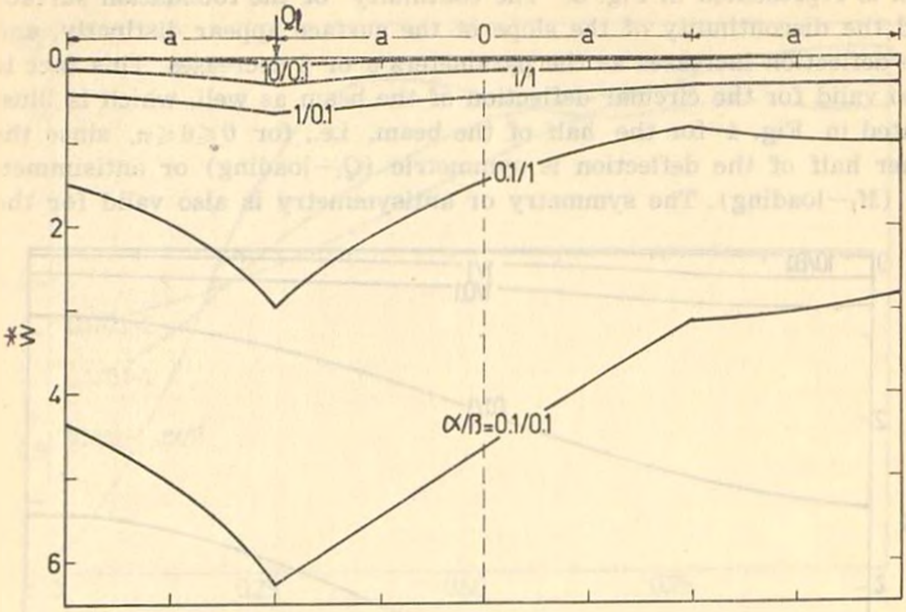

Fig. 3 (a). The surface of the foundation (Q_l—loading) for $\theta=0$ and $\theta=\pi$.

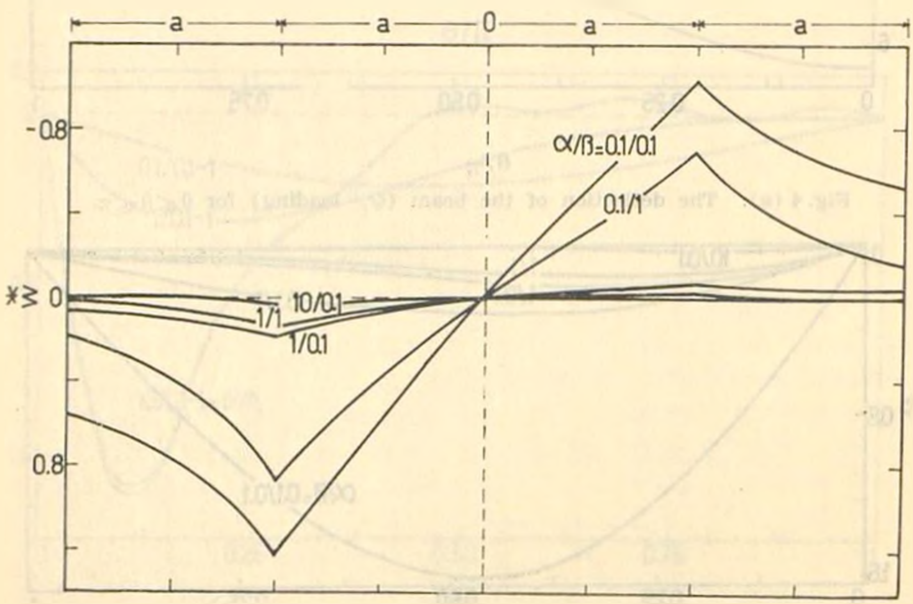

Fig. 3 (b). The surface of the foundation (M_l—loading) for $\theta=0$ and $\theta=\pi$.

tion is represented in Fig. 3. The continuity of the foundation surface and the discontinuity of the slope of the surface appear distinctly, and the deflection increases as the parameters α or β decreases. This fact is also valid for the circular deflection of the beam as well, which is illustrated in Fig. 4 for the half of the beam, i.e., for $0 \leqslant \theta \leqslant \pi$, since the other half of the deflection is symmetric (Q_l—loading) or antisimmetric (M_l—loading). The symmetry or antisymmetry is also valid for the

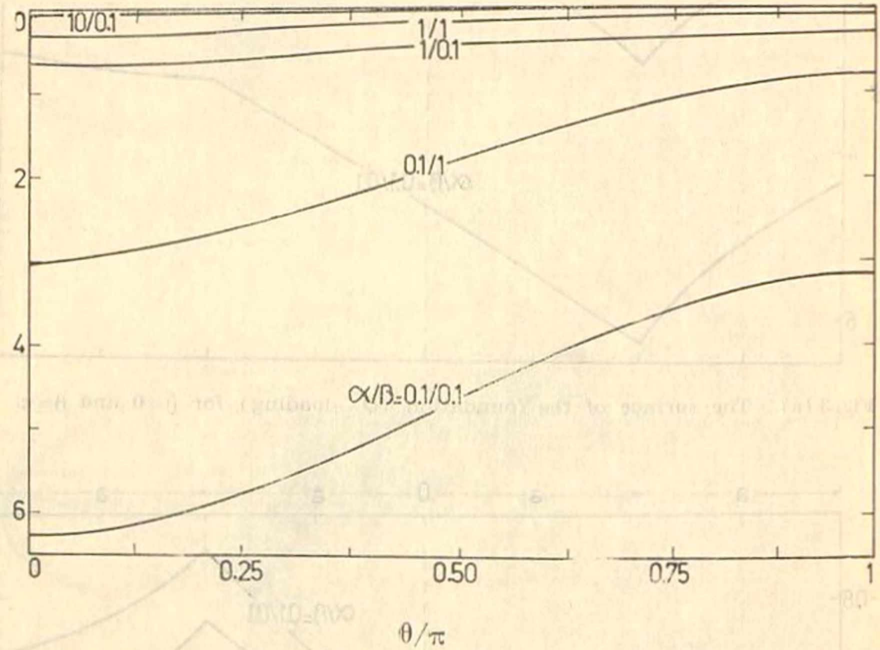

Fig. 4 (a). The deflection of the beam (Q_l—loading) for $0 \leqslant \theta \leqslant \pi$.

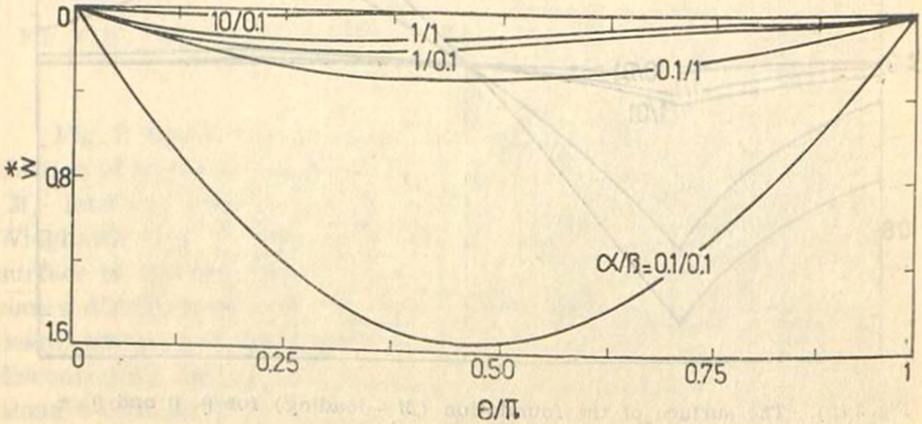

Fig. 4 (b). The deflection of the beam (M_l—loadiny) for $0 \leqslant \theta \leqslant \pi$.

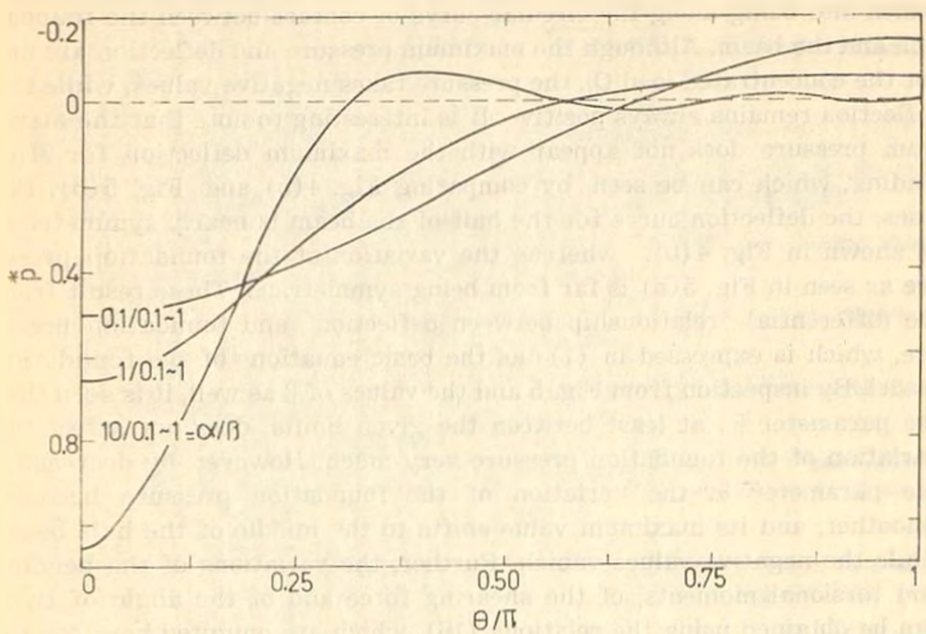

Fig. 5 (a). The foundation pressure under the beam (Q_l—loading) for $0 \leqslant \theta \leqslant \pi$.

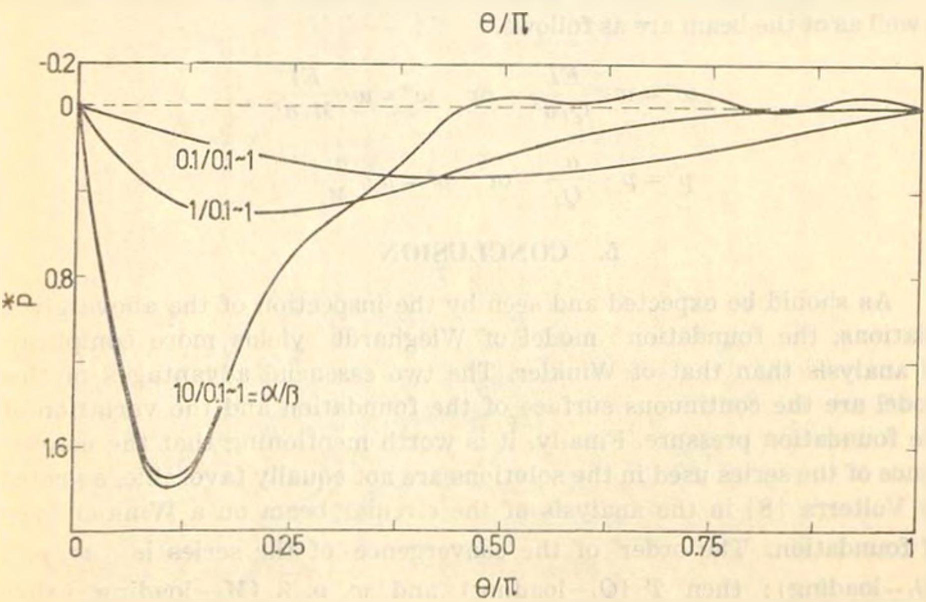

Fig. 5 (b). The foundation pressure under the beam (M_l—loading) for $0 \leqslant \theta \leqslant \pi$.

represantation of the foundation pressure which is shown in Fig. 5 and comes into being along the circular curve of contact between the foundation and the beam. Although the maximum pressure and deflection are under the concentrated load Q_l, the pressure takes negative values, while the deflection remains always positive. It is interesting to note that the maximum pressure does not appear with the maximum deflection for M_l- loading, which can be seen by comparing Fig. 4(b) and Fig. 5(b). Besides, the deflection curve for the half of the beam is nearly symmetrical as shown in Fig. 4(b), whereas the variation of the foundation pressure as seen in Fig. 5(b) is far from being symmetrical. These result from the differential relationship between deflection and foundation pressure, which is expressed in (1) as the basic equation of the foundation model. By inspection from Fig. 5 and the values of β as well, it is seen that the parameter β, at least between the given limits, does not affect the variation of the foundation pressure very much. However, by decreasing the parameter α, the variation of the foundation pressure becomes smoother, and its maximum value shifts to the middle of the half beam, while the negative values vanish. Further, the variations of the bending and torsional moments, of the shearing force and of the angle of twist can be obtained using the relations (16), which are ommited here for the sake of brevity.

Note that the illustrated dimensionless quantities of the foundation as well as of the beam are as follows,

$$w^* = w \cdot \frac{EI}{Q_l a^3} \quad \text{or} \quad w^* = w \cdot \frac{EI}{M_l a^2},$$

$$p^* = p \cdot \frac{a}{Q_l} \quad \text{or} \quad p^* = p \cdot \frac{a^2}{M_l}.$$

5. CONCLUSION

As should be expected and seen by the inspection of the above given relations, the foundation model of Wieghardt yields more complicated analysis than that of Winkler. The two essencial advantages of this model are the continuous surface of the foundation and the variation of the foundation pressure. Finally, it is worth mentioning that the convergence of the series used in the solutions are not equally favorable, as noted by Volterra [8] in the analysis of the circular beam on a Winkler type of foundation. The order of the convergence of the series is : w, p, β (Q_l-loading); then T (Q_l-loading) and w, p, β (M_l-loading); then M (Q_l-loading); and T (M_l-loading); then Q (Q_l-loading) and M (M_l-loading). The least favorable series is that of Q (M_l-loading).

REFERENCES

1. WIEGHARDT, K., Über den Balken auf nachgiebiger Unterlage, Z. ang. Math. Mech. **2**, 165 - 184 (1922).

2. SCHIEL, F., Der schwimmende Balken, Z. ang. Math. Mech. 22, 255 - 262 (1942).

3. LOOF, H. W., The theory of the coupled spring foundation as applied to the investigation of structures supported on soil, Heron **3**, 29 - 49 (1965).

4. CELEP, Z., Semi-infinitily long beam on a Wieghardt type elastic foundation, Bull. Tech. University Istanbul **30**, 69 - 82 (1977).

5. YLINEN, A. and MIKKOLA, M., A beam on a Wieghardt type elastic foundation, Int. J. Solids Structures **3**, 617 - 633 (1967).

6. SMITH, T. E., Buckling of a beam on a Wiegharat type elastic foundation, Z. ang. Math. Mech. **49**, 641 - 645 (1969).

7. ANDERSON, G. L., The influence of a Wieghardt type elastic foundation on the stability of some beams subjected to distributed tangential forces, J. Sound Vibration **44**, 103 - 118 (1976).

8. VOLTERRA, E., Deflection of circular beams resting on elastic foundation obtained by methods of harmonic analysis, J. Appl. Mech. **22**, 227 - 232, (1953).

9. BECHERT, H., Zur Berechnung der Kreisringfundamente auf elastischer Unterlage, Beton und Stahlbetonbau **53**, 156 - 158 (1958).

10. MCLACHNAN, N. W., Bessel functions for engineers, 2nd Edn. Clarendon Press, Oxford (1961).