Viskoelastik Plakların Basit Kayma Etkisindeki Davranışları

Behaviour Of Viscoelastic Plates Under Pure Shear

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Bu çalışmada kenarları boyunca basit kayma gerilmeleri etkisinde bulunan viskoelastik plakların davranışları incelenmiştir. İlkel eğrilikli plakların sehimleri uzay ve zaman değişkenleri cinsinden elde edilmiştir. İlgili viskoelastik plâk denklemi Laplace dönüşümü ve Galerkin yönteminden yararlanılarak çözülmüştür. Kayıcı ve ankastre mesnetli dikdörtgen plâklar için örnekler verilmiştir. Viskoelastik malzeme özellikleri için Maxwell, Kelvin - Voigt ve standart lineer katı modelleri kullanılmıştır.

In this paper, the behaviour of plates of linear viscoelastic material under pure shear has been investigated. The deflection of the plate with initial curvature has been determined as a function of space and time variables by solving the related viscoelastic plate equation using Laplace transformation over the time domain together with the method of Galerkin. Examples are presented for rectangular plates with simply supported and camped edges. Maxwell, Kelvin - Voigt and standard linear solid models are used to describe viscoelastic behaviour.

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I — Introduction

Creep and relaxation are the two important behaviour of viscoelastic materials. Creep is defined as slow and continuous material deformation under constant stress. While relaxation is reduction in stress under constant strain. Various investigations have been carried out on the effect of creep in engineering problems. [1], [2], [3], [4].

Creep problems of viscoelastic plates and shells have been the subject of many investigators. Mase [5] and Pister [6] studied the bending of linear viscoelastic plates. Lin [7] solved the problem of creep deflection of viscoelastic plate with initial curvature under uniform edge compression. Lin used Laplace transformation technique to solve the plate equation. The same problem has also been investigated by DeLeeuw and Mase [8] both with and without initial curvature. DeLeeuw [9] applied the same method to circular viscoelastic plates subjected to in - plane forces.

In this paper, rectangular plates with initial curvature subjected to simple shearing forces along the edges will be analized.

II. Formulation of the Problem

The governing equations are developed using the cartesian coordinates. The Standard positive sign convertion for this coordinate system is used for the stresses, strains and diplacements (Figure 2.1).

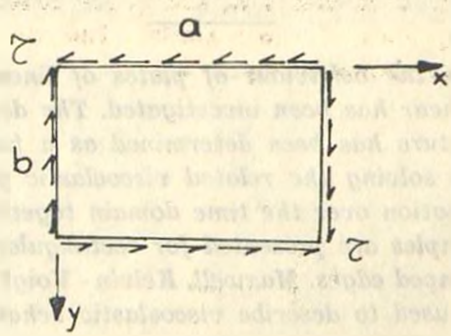

Figure 2.1

2.1. Assumptions

The study will be conducted under the following assumptions :

A1 — Inertia effects are neglected and a quasi - static analysis is investigated.

A2 — There are no body forces.

A3 — Plate is made of homogeneous, isotropic, incompressible and linear viscoelastic material.

A4 — The thickness (h) of the plate is much smaller than the typical plate dimension.

A5 — Bernoulli - Navier hypothesis is valid for the plate.

2.2. *Stress - strain relations*

The stress - strain relations of incompressible linear viscoelastic materials can be given as

$$\underset{\sim}{P}\, S_{ij} = 2\underset{\sim}{Q}\, e_{ij} \qquad (2.1.)$$

In these equations e_{ij} and S_{ij} are the deviatoric strain and stress tensors respectively. $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are linear differential operators of the form

$$\begin{aligned} \underset{\sim}{P}(p) &= p_0 + p_1 p + \ldots + p_m p^m \\ \underset{\sim}{Q}(p) &= q_0 + q_1 p + \ldots + q_n p^n \end{aligned} \qquad (2.2.)$$

where $p = \partial/\partial t$. The coefficients p_m and q_n are constants which represent the physical properties of the material. The viscoelastic models used in the analysis are shown in Figure 2.2.

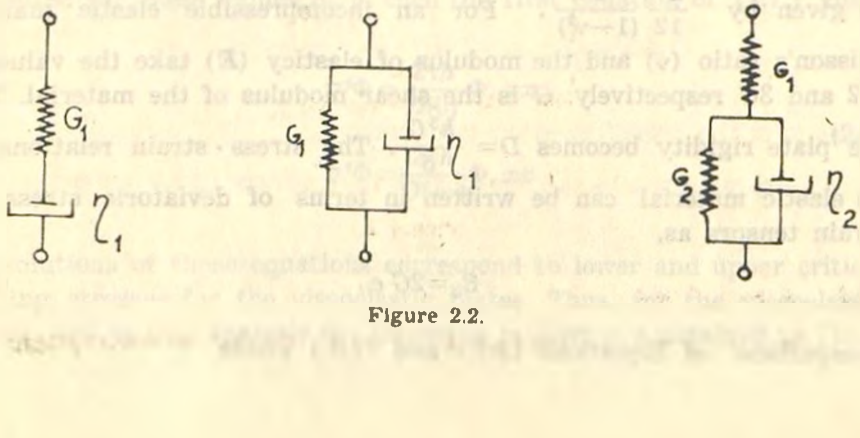

Figure 2.2.

Maxwell Model	Kelvin - Voigt Model	Standard Linear Solid
$p_0=1$	$p_0=1$	$p_0=1$
$p_1=\frac{\eta_1}{G_1}$	$p_1=0$	$p_1=\frac{\eta_2}{G_1+G_2}$
$q_0=0$	$q_0=G_1$	$q_0=\frac{G_1G_2}{G_1+G_2}$
$q_1=\eta_1$	$q_1=\eta_1$	$q_1=\frac{G_1\eta_2}{G_1+G_2}$

2.3. Governing Equation of the Viscoelastic Plate

Using the stress - strain relations given by Equation (2.1.) and making use of the classical plate theory, the following equation can be obtained [10] as,

$$D(p)\,\nabla^4 w(x\,,y\,,t)=2h\,\tau\,w_{,xy} \qquad (2.3.)$$

In this equation, $D(p)$ is a differential operator which corresponds to the bending rigidity of an incompressible linear viscoelastic plate given by the relation,

$$D(p)=\frac{h^3}{3}\,\frac{Q\,(p)}{P\,(p)} \qquad (2.4)$$

∇^4 is the biharmonic operator over space variables x and y and «,» denotes the partial differentiation. w isc the out - of - plane displacement of the plate middle surface. The bending rigidity of an elastic plate can be given by $\frac{h^3}{12}\,\frac{E}{(1-\nu^2)}$. For an incompressible elastic material Poisson's ratio (ν) and the modulus of elasticy (E) take the values of 1/2 and 3G respectively. G is the shear modulus of the material. Then the plate rigidity becomes $D=\frac{h^3G}{3}$. The stress - strain relations for an elastic material can be written in terms of deviatoric stress and strain tensors as,

$$S_{ij}=2G\,e_{ij} \qquad (2.5.)$$

Comparison of Equations (2.1.) and (2.5.) yields

$$G = \frac{\underset{\sim}{Q}}{\underset{\sim}{P}}$$

Thus, the bending rigidity of a viiscoelastic plate can be obtained from the elastic plate by substituting the values of shear modulus in terms of viscoelastic operators $\underset{\sim}{Q}$ and $\underset{\sim}{P}$.

2.4. *Stability Problem*

The solution for the viscoelastic plate equation (2.3.) can be taken as

$$w\,(x\,,y\,,y) = T(t)\,\Phi\,(x\,,y) \tag{2.6.}$$

Substitution of (2.6.) in (2.3.) yields

$$\nabla^4\Phi(x,y) = \frac{2}{C}\tau h\Phi(x,y),\ xy$$

and

$$[D(p) - C]T(t) = 0 \tag{2.7}$$

The solution of the second equation can be obtained as $T(t) = A\,e^{pt}$ if the operators are considered as algebraic functions [11]. Since p represents differentiation with respect to time, then, the stability criteria can be determined from the increase in the deflection rates. Thus, the values corresponding to $p=0$ and $p=\infty$ give the lower and upper bounds for the instantaneous buckling stresses. If the bending rigidity is calculated for these values of p, then the first equation of (2.7.) yields

$$\nabla^4\Phi = \frac{2\tau h}{D(0)}\Phi,\ xx$$

and

$$\nabla^4\Phi = \frac{2\tau h}{D(\infty)}\Phi,\ xx \tag{2.8}$$

The solutions of these equations correspond to lower and upper critical buckling stresses for the viscoelastic plates. Thus, for the viscoelastic models used in this analysis the following relation are obtained as [10]:

Maxwell Model

$$\tau_{cr}\ (\text{lower}) = 0 \qquad \tau_{cr}(\text{upper}) = \frac{k\pi^2 h^2 G_1}{3b^2}$$

Kelvin Model

$$\tau_{cr}(\text{lower}) = \frac{k\pi^2 h^2 G_1}{3b^2} \qquad \tau_{cr}(\text{uppper}) = \infty \tag{2.9}$$

Standard Linear Solid

$$\tau_{cr}(\text{lower}) = \frac{k\pi^2 h^2}{3b^2}\ \frac{G_1\, G_2}{G_1 + G_2} \qquad \tau_{cr}(\text{upper}) = \frac{k\pi^2 h^2 G_1}{3b^2}$$

In these equation k depends on the plate geometry and the boundary conditions along the edges of the plate and can be found in [12].

2.5. *Viscoelastic Plate with Initial Curvature*

In many instances, the middle plane of the plate may have some kind of small inperfections. This can be represented as an initial deflection small compared to the thickness of the plate. Thus the plate equation takes the form

$$\underline{D}(p)\Delta^4 w_1(x,y,t) = 2\tau h[w_0(x,y) + w_1(x,y,t)],xy \tag{2.10}$$

Here $w_1(x\,,y\,,t)$ is the additional deflection due to bending effect.

III — Solution of the Problem

The partial differential equation given by (2.10.) can be solved using the Laplace transformation. Then the above equation gives

$$\underline{\bar{D}}(s)\Delta^4 \bar{w}_1(x,y,s) = 2\,\tau h\left(\frac{w_0}{s} + \bar{w}_1(x,y,s)\right).xy \tag{3.1}$$

where s is the transformation parameter.

Galerkin's method can be applied to equation (3.1.) by considering

$$\bar{w}_1(x,y,s) = \sum_{i=1}^{n}\ \sum_{j=1}^{m}\bar{B}_{ij}(s)\psi_i(x)\phi_j(y)$$
$$w_0(x,y) = \sum_{i=1}^{n}\ \sum_{j=1}^{m}A_{ij}\psi_i(x)\phi_j(y) \tag{3.2}$$

for the addition and initial deflections respectively. In these equations $\bar{B}_{ij}(s)$'s are unknown coefficients to be determined from the initial deflection using the Fourier analysis. ψ_i 's and ϕ_j 's are coordinate functions satisfying the boundary conditions along the edges of the plate. For simplicity, these functions can be chosen as the eigenfunctions of a transversly vibrating beam with similar edge conditions. These coordinate functions are chosen as

$$\left.\begin{aligned} \psi_i &= \sqrt{\frac{2}{a}} \sin \lambda_i x \qquad \lambda_i = \frac{i\pi}{a} \\ \phi_j &= \sqrt{\frac{2}{b}} \sin \lambda'_j y \qquad \lambda'_j = \frac{a}{b} \lambda_i \end{aligned}\right\} \tag{3.3}$$

and

$$\left.\begin{aligned} \psi_i(x) &= \frac{1}{\sqrt{a}} [(\cos \lambda_i x - ch\, \lambda_i x) - \alpha_i (\sin \lambda_i x - sh \lambda_i x)] \\ \alpha_i &= \frac{\cos \lambda_i a - ch\, \lambda_i a}{\sin \lambda_i a - sh\, \lambda_i a} \qquad \lambda_i = \frac{i\pi}{a} \\ \phi_j(y) &= \frac{1}{\sqrt{b}} [(\cos \lambda'_j y - ch\, \lambda'_j y) - \alpha'_j (\sin \lambda'_j y - sh\, \lambda'_j y)] \\ \alpha'_j &= \frac{\cos \lambda'_j b - ch\, \lambda'_j b}{\sin \lambda'_j b - sh\, \lambda'_j b} \qquad \lambda'_j = \frac{a}{b} \lambda'_i \end{aligned}\right\} \tag{3.4}$$

representing the simply supported and clamped edges respectively.

Substitution of Equation (3.2.) together with (3.3.) into the equation (3.1.) and the application of Galerkin's method using the orthogonality properties of the coordinate functions gives

$$\bar{D}(s)\bar{B}_{kr}(s)[k^2 + r^2 \alpha^2] = \frac{8h\tau\alpha^3 b^2}{\pi^4} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{A_{ij}}{s} + \bar{B}_{ij}(s) \right) \frac{ikjr\,(1-(1)^{i+k})\,(1-(1)^{j+r})}{(k^2-i^2)\,(r^2-j^2)} \right\}$$

$$\left(\alpha = \frac{a}{b}\ ,\ k = 1,2,\ \ldots,\ n \text{ and } r = 1,2,\ \ldots\ldots,\ m \right) \tag{3.5}$$

for simply supported plate [10].

For the case of clamped plate, using (3.4.) and with similar manipulations

$$\bar{D}(s)\Big\{\bar{B}_{kr}(s)\Big[\lambda_k^4+\lambda_r'^4+2\Big(-\lambda_k^2\alpha^2+2\lambda_k\frac{\alpha_k}{a}\Big)\Big(-\lambda_r'^2\alpha_r'^2+2\lambda'_r\frac{\alpha'_r}{b}\Big)+$$

$$2\sum_{i=1}^{n}\sum_{j=1}^{m}\bar{B}_{ij}(s)\,\frac{16\,\lambda_i^2\,\lambda_k^2\,\lambda_j'^2\,\lambda_r'^2\,(\lambda_i\,\alpha_i-\lambda_k\,\alpha_k)\,(1+(-1)^{i+k})}{ab(\lambda_i^4-\lambda_k^4)\,(\lambda_j'^4-\lambda_r'^4)}$$

$$\frac{(\lambda'_j\,\alpha'_j-\lambda'_r\alpha'_r)\,(1+(-1)^{j+r})}{}\Big\}=2h\tau\sum_{i=1}^{n}\sum_{j=1}^{m}\Big\{\Big(\frac{A_{ij}}{s}+\bar{B}_{ij}(s)\Big)$$

$$\frac{16\,\lambda_i^2\,\lambda_k^2\,\lambda_j'^2\,\lambda_r'^2\,((-1)^{i+k}-1)\,((-1)^{j+r}-1)}{ab\,(\lambda_i^4-\lambda_k^4)\,(\lambda_j'^4-\lambda_r'^4)}\Big\} \qquad (3.6)$$

is obtained.

The sets of algebraic equations in (3.5.) and (3.6.) can be solved for $\bar{B}_{ij}(s)$'s. Then, the inverse Laplace transformation is applied in order to obtain $B_{ij}(t)$'s [10]. Consequently, the final plate deflection can be computed from the relation

$$w(x,y,t)=w_0(x,y)+w_1(x,y,t). \qquad (3.7.)$$

IV — Examples

Simply Supported Plate :

A parabolic surface is considered to define the initial curvate of the plate. Fourier analysis yields the coefficients of the initial deflection as

$$A_{ij}=\int_0^b\int_0^a w_{or}\Big(-\frac{4x^2}{a^2}+\frac{4x}{a}\Big)\Big(-\frac{4y^2}{b^2}+\frac{4y}{b}\Big)\psi_i(x)\phi_j(y)\,dx\,dy \qquad (4.1)$$

where w_{OR} is the initial deflection of the plate center.

Also $G_1=140647$ kg/cm², $G_2=365682$ kg/cm², $\eta_1=281294$ (kg−hr)/cm² and $\eta_2=36568{,}2$ (kg−hr)/cm² are chosen for the constants used in visco-elastic models [9].

Using the information given above, numerical calculations are carried out and presented below for different viscoelastic models.

Maxwell model : In this case, the upper bound for the critical shear stress becomes

$$\tau_{cr} = \frac{k}{3} \left(\frac{\pi h}{b} \right)^2 \cdot G_1 \tag{4.2}$$

The numerical calculations are carried out for $\tau = \tau_{cr}/4$, $\tau = \tau_{cr}/2$ and $\tau = 3\tau_{cr}/4$. For the solution, two and five terms in the series are used and the convergence of the result is found to be satisfactory. The plate deflection at section $y = b/4$ and $y = b/2$ are presented as a function of time in Figure 4.1. and Figure 4.2. respectively for a shear stress

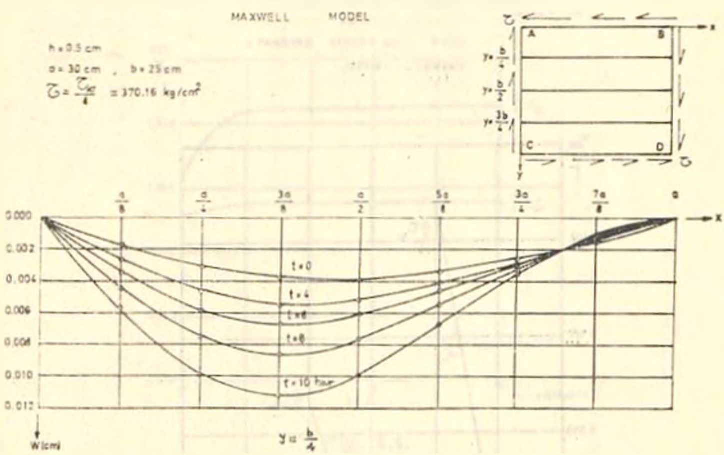

Fig. 4.1.

$\tau = \tau_{cr}/4$. The deflection of the plate center is shown as a function of time in Figure 4.3. for different values of shear stress. As it is seen from these curves, there is no bound on the deflection.

Kelvin - Voigt Model : For this model, the lower bound for the critical shear stresses can be calculated from the equation (2.9.). The plate center deflections for this model in shown in Figure 4.4a. There are bounds for the plate center deflections, which is parallel to the behaviour of Kelvin model under creep.

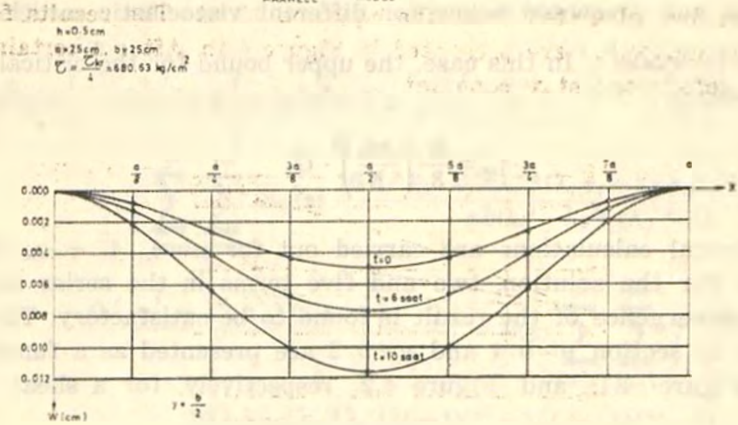
MAXWELL MODEL
h=0.5cm
a=25cm , b=25cm
$\tau = \frac{\tau_{kr}}{4} = 680.63$ kg/cm^2
0.000
0.002
0.004
0.006
0.008
0.010
0.012
W(cm)
$y = \frac{b}{2}$
$\frac{a}{8}$ $\frac{a}{4}$ $\frac{3a}{8}$ $\frac{a}{2}$ $\frac{5a}{8}$ $\frac{3a}{4}$ $\frac{7a}{8}$ a
x
t=0
t = 6 saat
t = 10 saat

Fig. 4.2.

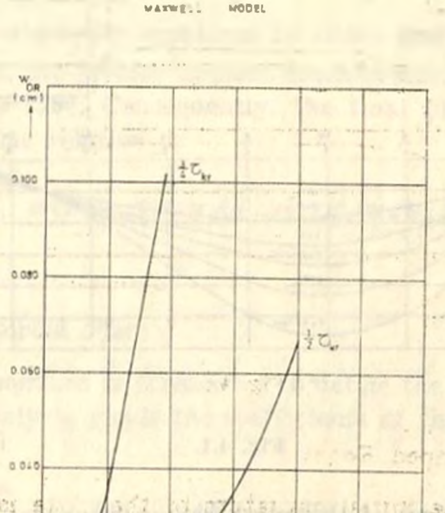
MAXWELL MODEL
W_{OR} (cm)
0.100
0.080
0.060
0.040
0.020
0.000
$\frac{1}{2}\tau_{kr}$
$\frac{1}{2}\tau_{kr}$
$\frac{1}{4}\tau_{kr}$
0 2 4 6 8 10
t (saat)
h=0.5cm
a=25cm , b=25cm
$\tau_{kr} = 2722.52$ kg/cm^2

Fig. 4.3.

Standard Linear Solid : The lower and upper bounds for the critical stresses can be calculated from the equation (2.9.). The results for the plate center deflection are presented in Figure 4.4b. After a certain time the plate deflections stay constant.

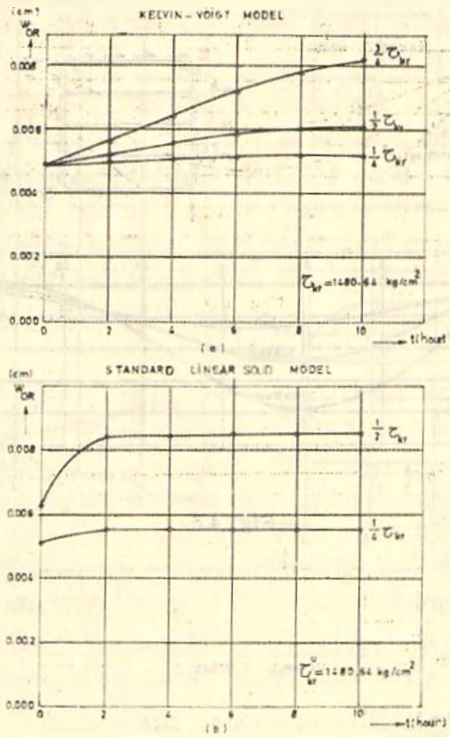

Fig. 4.4.

Plate with Clamped Edges :

In this case, the initial curvature of the plate is considered as

$$w_o(x,y) = w_{oR}\,\frac{1}{4}\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi y}{b}\right) \tag{4.3}$$

and from the Fourier analysis, the coefficients become

$$A_{ij} = \int_0^b \int_0^a w_o(x,y)\psi_i(x)\phi_j(y)\,dx\,dy \tag{4.4}$$

Using this information, numerical calculations are carried out for Maxwell, Kelvin - Voigt and standard linear solid models. Similar results are obtained and presented in Figures (4.5.), (4.6.), (4.7.) and (4.8.).

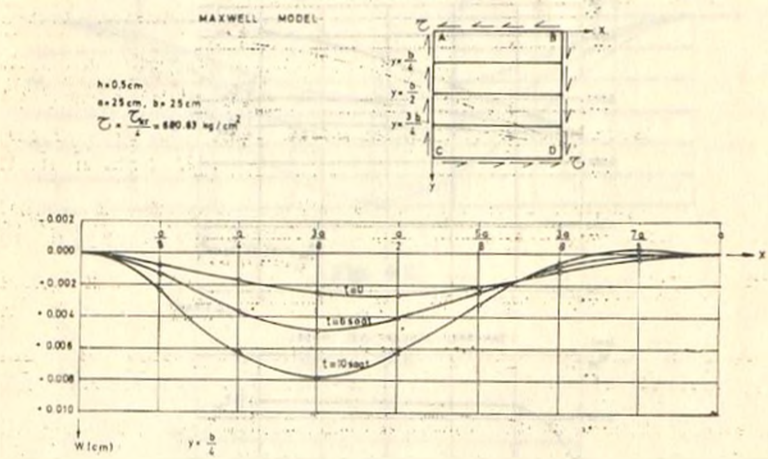

Fig. 4.5.

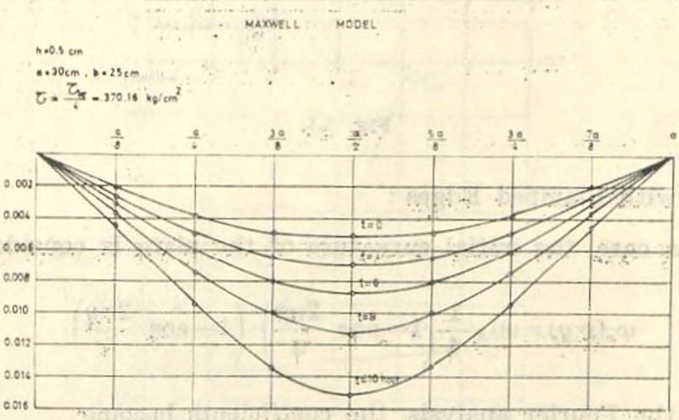

Fig. 4.6.

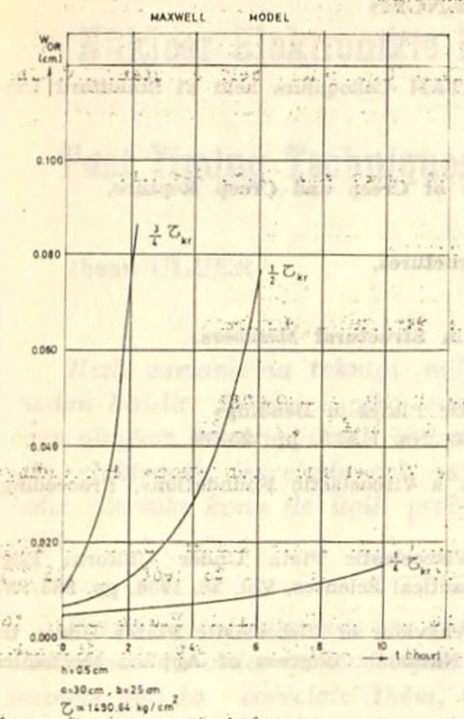

Fig. 4.7.

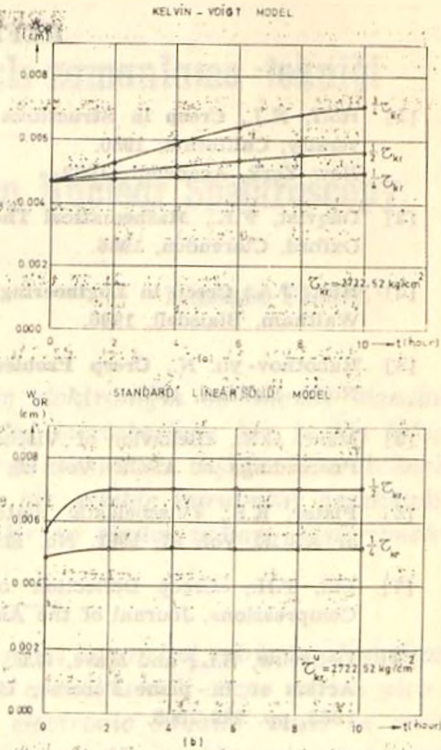

Fig. 4.8.

V — Conclusions

As a result, a viscoelastic plate of Maxwell material will have a creep buckling at any load greater than zero. While instantaneous buckling stresses are equal to that of elastic plates.

For a Kelvin - Voigt model, there will be no creep buckling for a load smaller than the instantaneous buckling load which is also equal to the elastic buckling load. In the case of standard linear solid the instantaneous buckling load is smaller than the elastic buckling load.

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