Internally Damped Vibration of Systems

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Abstract :

Structural or internal damping is defined, different types of damping mechanisms are classified and explained. Complex represantation is used as a mathematical model. It is well known that the modes for undamped vibration are orthogonal for an elastic structure The situation is searched for damped vibration and it is found that the modes are not orthogonal for internally damped systems. This is because for damped vibration, Young's modulus is a function of frequency and differs from state to state. The comparision of the viscous and internal damping characters is made and the relation between damping factors which correspond to both damping, is found.

Sistemlerin iç sürtünmelerle titreşimlerinin sönümü :

Özet :

İç sürtünmeli titreşim sönümü tarif edildi, ve bunu meydana getiren değişik tip mekanizmalar sınıflandırılıp, izah edildi. Kompleks değişkenlerle çözüm bir matematik model olarak kullanıldı. Elastik sistemlerin sönümsüz titreşimlerinde, modların ortogonal olduğu bilinir. Bu durum sönümlü titreşimler için araştırıldı ve sönümlü titreşimin modlarının ortogonal olmadığı görüldü. Buna sebep, sönümlü titreşim halinde Young modülünün frekansın bir fonksiyonu olduğu ve değişik frekanslar için değişik değerler aldığıdır. Viskoz ve iç sürtünmeli titreşimlere ait sönüm karekterleri mukayese edildi ve bu iki hale ait sönüm faktörleri arasındaki bağıntı bulundu.

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Introduction :

The word damping has been used for many years to denote the noise reducing procedures. The mechanical meaning of this word is converting the mechanical vibration energy of solids into the form of other kinds of energy. In other words, it is a removal of energy from a vibratory system. The energy lost is either transmitted away by some mechanisms or dissipated within the material. Investigations on the damping of materials and its application in engineering was started about 200 years ago. First in 1784, Coulomb recognized that the damping at low stresses may be different from those at high stresses. Then he proved that the damping is not only caused by air friction, but also by internal losses in the materials. After him, many investigators have been studying the viscosity of metals, its non-linear behaviour and the effect of stress amplitude, frequency and temperature on the vibration of solids. Voigt worked on the cyclic bending and the hysteresis loop. In the first decades of the twentieth century, investigations were initiated on the possible relationship between damping and fatigue of materials. After 1950, damping has been increasingly important for studying of noise reduction. In the past ten years research and engineering interest in damping of viscoelastic materials has increased.

The control of noise and vibration by the application of damping became standart practise in industries. Deadners have been used on the car bodies to reduce the noise levels inside the car. Many manufactures have been applying «Damping Tapes» for noise control.

Before, noisy operation and resonancy have always been problem areas. These effects could usually be minimized in the previous years by seperation of the natural frequency of the system and the exciting frequency But sometimes such a seperation of frequencies is besoming very difficult for the materials which are having many resonances. Also, random excitation, either of mechanical or accoustical origin, become a common problem. For example, a jet noise, generally contains most of the natural frequencies in airplane structures. In this case, there is no way to separate the excitation and natural frequencies. The maximization of the damping within a structural system is most useful way in controlling resonance and noise problem.

Damping Behavior :

Under cyclic loading conditions, strain is not a linear function of stress. Materials do not behave in a perfectly elastic manner at very low stresses. In all cases, materials or structural systems that dissipate energy under cyclic load, display one phenomenon in common; the cyclic load-deformation or stress-strain curve form a hysteretic loop. The area in this closed loop is proportional to the energy absorbed.

Many different types of damping mechanisms have been classified by investigators. This is as;

A — Anelasticity effects

B — Linear damping mechanisms associated with dislocation

C — Static hysteresis

A large variety of anelastic mechanisms has been identified in metals. Important types of these are;

a — Macrothermoelasticity effects

b — Microthermoelasticity effects

c — Grain boundary effects

d — Eddy current effects

All types of these mechanisms effect the damping behavior. But the most important one is the thermoelastic effect and it became a common mechanism for bending vibration of metals.

In bending vibration, the material that is on the outer or convex side is expanded and cooled while that on the concave side is compressed and raised in temperature. Heat flow will occur across the material and there will be some energy loss.

Complex Representation :

When we have a material which is subject to time-dependent variation of stress and strain, the fundamental deformation is no longer related to stress by a simple constant of proportionality. Or, when we have a model which is subjected to harmonic loading conditions, strain is not a linear function of stress. Mostly, the governing equation is a linear partial differential equation of arbitrary order of the from

$$\left(A_0 + A_1 \frac{\partial}{\partial t} + A_2 \frac{\partial^2}{\partial t^2} + \cdots + A_n \frac{\partial^n}{\partial t^n}\right) \sigma = \left(B_0 + B_1 \frac{\partial}{\partial t} + B_2 \frac{\partial^2}{\partial t^2} + \cdots + B_n \frac{\partial^n}{\partial t^n}\right) \varepsilon \quad (1)$$

where A_n and B_n are constants, t is time, σ is stress and ε is strain. For sinusoidal time dependent stress and strain as;

$$\sigma=\sigma_0 e^{i\omega t} \quad , \quad \varepsilon=\varepsilon_0 e^{i\omega t} \tag{2}$$

and substituting these relations into eqn. (1), we obtain

$$[A_0+(i\omega)A_1+(i\omega)^2 A_2+\cdots+(i\omega)^n A_n]\sigma=[B_0+(i\omega)B_1+(i\omega)^2 B_2+\cdots+(i\omega)^n B_n]\varepsilon$$

which can be easily written as

$$E^*=\frac{\sigma}{\varepsilon}=E(\omega)+i\,E'(\omega) \tag{3}$$

where $E(\omega)$ and $E'(\omega)$ are functions of frequency. That is to say, the ratio of stress to strain in the material may be represented not by a real number but by a complex quantity which is Young's Complex modules. When we have a material which is stressed in shear, the complex ratio of stress to strain may be written as

$$G^*=G(\omega)+i\,G'(\omega) \tag{4}$$

where $G(\omega)$ and $G'(\omega)$ are functions of frequency. Eqns. (3) and (4) can be written in the following form

$$E^*=E(\omega)[1+i\,\delta_E(\omega)]$$

$$G^*=G(\omega)[1+i\,\delta_G(\omega)] \tag{5}$$

where (δ_E) is the ratio of imaginary part to real part of complex Young's modulus and called as damping factor for bending vibration, (δ_G) is the ratio of imaginary part to real part of complex shear modulus and called as damping factor for shear vibration.

Similarly the Bulk modulus will be in the form as

$$B^*=B(\omega)[1+i\,\delta_G(\omega)] \tag{6}$$

There is a relation between these three material moduli as

$$E^*=\frac{9\,B^*G^*}{(3\,B^*+G^*)} \tag{7}$$

For viscoelastic materials, $\delta_B \ll \delta_G$ and $G/B \ll 1$. With these approximations, we find that;

$$E\cong 3\,G \quad \text{and} \quad \delta_E\cong\delta_G \tag{8}$$

Orthogonality of eigenvectors of internally damped vibration :

It is known that linear dynamic systems without damping give normal modes. Now, we will apply this property to the internally damped systems and see if it is satisfied.

The governing equations for a beam are

$$\frac{dV(x)}{dx} = -p(x) \quad , \quad \frac{d\beta(x)}{dx} = -\frac{M(x)}{E^*I} \tag{9}$$

$$\frac{dM(x)}{dx} = V(x) \quad , \quad \frac{dw(x)}{dx} = \beta(\omega)$$

Let us consider two different states of vibration for which the variables are

V_i, M_i, β_i, w_i State I

V_j, M_j, β_j, w_j State II

The first equation of eqn. (9), for state I is

$$\frac{d\,V_i(x)}{dx} + p_i(x) = 0$$

Multiplying both terms by w_j and integrate along the beam, we get

$$\int_0^L \frac{dV_i}{dx} \cdot w_j\,dx + \int_0^L p_i\,w_j\,dx = 0 \tag{10}$$

Integrating eqn. (10) by parts and using eqn. (9), we obtain

$$w_j V_i \Big|_0^L - M_i \beta_j \Big|_0^L + \frac{1}{E_j^* I} \int_0^L M_i M_j\,dx + \int_0^L p_i \cdot w_j\,dx = 0 \tag{11}$$

A similar relation can be found as in the following form

$$w_i V_j \Big|_0^L - M_j \beta_i \Big|_0^L + \frac{1}{E_i^* I} \int_0^L M_j M_i\,dx + \int_0^L p_j\,w_i\,dx = 0 \tag{12}$$

From eqns. (11) and (12), we obtain

$$w_j V_i \Big|_0^L - M_i \cdot \beta_j \Big|_0^L - w_i \cdot V_j \Big|_0^L + M_j \beta_i \Big|_0^L + \int_0^L \left(\frac{1}{E_j{}^* I} M_i M_j - \frac{1}{E_i{}^* I} M_j M_i \right) dx +$$

$$+ \int_0^L (p_i w_j - p_j w_i)\, dx = 0 \qquad (13)$$

In this equation, first four terms are boundary conditions and assuming both states I and II satisfy the same boundary conditions, then eqn (13) becomes

$$\int_0^L (p_i w_j - p_j w_i)\, dx = \int_0^L \left(\frac{1}{E_j{}^* I} M_i M_j - \frac{1}{E_i{}^* I} M_j M_i \right) dx \qquad (14)$$

For undamped vibration, E^*I does not change from state I to state II. Thus

$$E_i{}^* I = E_j{}^* I = E^* I \qquad (15)$$

And the expression on right hand side of eqn. (14) is symmetric with respect to the endices i and j.

Let P be as;

$$p = -\rho h \frac{\partial^2 w}{\partial t^2} \qquad (15)$$

and

$$w(x,t) = w(x)\, e^{i\omega t} \qquad (17)$$

then we obtain

$$(\omega_i^2 - \omega_j) \int_0^L w_i w_j\, dx = 0$$

For the case of $\omega_i \neq \omega_j$, we find that

$$\int_0^L w_i \cdot w_j\, dx = 0 \qquad (18)$$

This is the orthogonality condition for undamped free vibration of beam. But when we have structural damping involved with the motion, we will have difficulties to get the normal modes. Because, the Young's modulus E^* will display different values for different frequencies with structural damping included. In other words, the Young's modulus is no more constant but it is dependent on frequency. So, when we take structural damping into account, then

$$E_i^* I \neq E_j^* I \tag{19}$$

and equation (18) is not equal to zero. Therefore, when $\omega_i \neq \omega_j$

$$\int_0^L w_i \cdot w_j \, dx \neq 0 \tag{20}$$

Thus, we now have an important result which can be stated as; The eigenvectors of internally damped vibration of a beam are not orthogonal, since Young's modulus E^* varies with frequency.

The comparision of the viscous and the structural damping characters :

In general, analytical solution of damping in vibrating systems are solved with damping forces proportional to velocity. More recently, there has been introduced the concept of a damping force proportional to amplitude. In both cases, the damping forces are in phase with the velocity of vibration. Now, we can compare these two different kinds of damping in the following way.

It is well known that for the free vibration of a simple degree of freedom damped system sets up the differential equation of motion in the form

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{21}$$

where m : The mass of the system

c : Viscous damping coefficient

k : Spring stiffness

Structural damping appears to be in phase with the velocity but proportional to the amplitude while viscous damping force is proportional to the velocity.

The equation of the motion for internally damped system will be in the form

$$m\ddot{x}+k(1+i\,\delta)x=0 \tag{22}$$

where δ : Internal damping coefficent

Here, a complex stiffness, $k(1+i\,\delta)$, serves to represent both components of the force proportional to displacement, x. One component kx is the usual spring force. The other component, $ik\delta x$ is in phase with the velocity, this is the damping force:.

The solution of equation (21) has a known solution as

$$x(t)=e^{-c/2mt}\,(A_1\cos\omega_{n_1}t+B_1\sin\omega_{n_1}t) \tag{23}$$

where ω_{n1} : Natural frequency of the damped motion. and is given by

$$\omega_{n_1}=\sqrt{\left(\frac{k}{m}\right)-\left(\frac{c^2}{4m^2}\right)} \tag{24}$$

Here, ω_{n1} decreases as c increases.

In order to solve equation (22), let

$$x=x_o e^{rt} \tag{25}$$

Substituting this equation (25) into equation (22), we find

$$mr^2+k(1+i\,\delta)=0 \tag{26}$$

In this relation, either r is complex or δ must be zero. Let

$$r=r'+i\,r'' \tag{27}$$

Then

$$m(r'^2+i\,2\,r'r''-r''^2)+k(1+i\,\delta)=0$$

which gives two relations as

$$2\,mr'r''+k\,\delta=0$$

$$m(r'^2-r''^2)+k=0 \tag{28}$$

Form these two relations, we find

$$r'=\mp\left[\left(\frac{k}{2m}\right)(-1+\sqrt{1+\delta^2})\right]^{1/2} \tag{29}$$

$$r' = \mp \left[\left(\frac{k}{2m} \right) (1 + \sqrt{1+\delta^2}) \right]^{1/2} \tag{30}$$

In equation (29), negative r' must be taken into account in order to get the decay solution. Equation (30), gives the natural frequency of the internally damped system as

$$\omega_{n_2} = \left(\frac{k}{2m} \right)^{1/2} [1 + \sqrt{1+\delta^2}]^{1/2} \tag{31}$$

Thus the solution of the equation (22) will be

$$x(t) = e^{-r't} [A_2 \cos \omega_{n_2} t + B_2 \sin \omega_{n_2} t] \tag{32}$$

From the comparision of equations (24) and (31), we can see the difference between viscous and internal damping characters.

In a system which has viscous damping characters, gives a damped natural frequency which decreases with increasing in damping.

In a system which has internal damping character gives a damped natural frequency which increases with increase in damping.

In order to find a relation between viscous factor c and internal damping factor δ, we could compare the decay curves (envalope curves of the solutions).

The envalope of equation (23) is given by $e^{(-c/2m)t}$ and the envalope of the equation (32) is given by $e^{-r't}$

A measuring method of the damping in a free vibration is mostly known as the logaritmic decrement. The time interval for a cycle is given by $2\pi/\omega_n$ sec. Thus, the logaritmic decrement for the first case would be

$$\Delta_1 = \frac{e^{-\frac{c}{2m}t}}{e^{-\frac{c}{2m}\left(t + \frac{2\pi}{\omega_{n_1}}\right)}} = e^{\frac{\pi c}{m\omega_{n_1}}} = e^{\frac{\pi c}{m\sqrt{(k/m) - c^2/4m^2}}} \tag{33}$$

And the logarithmic decrement for the second case, would be

$$\Delta_2 = \frac{e^{-r't}}{e^{-r'\left(t+\frac{2\pi}{\omega_{n_2}}\right)}}$$

$$\Delta_2 = e^{\frac{2\pi}{\delta}[-1+\sqrt{1+\delta^2}]} \tag{34}$$

In order to get the same logaritmic decrement, the relation between δ and c would be

$$\frac{\pi c}{\sqrt{k/m - \frac{c^2}{4}}} = \frac{2\pi}{\delta}[-1+\sqrt{1+\delta^2}]$$

and

$$\frac{2c}{\sqrt{c_c^2 - c^2}} = \frac{2}{\delta}[-1+\sqrt{1+\delta^2}] = \frac{2\delta}{1+\sqrt{1+\delta^2}} \tag{35}$$

where $c_c = 2\sqrt{km}$, critical damping factor

Thus, the relation becomes

$$\frac{2\xi}{\sqrt{1-\xi^2}} = \frac{2\delta}{1+\sqrt{1+\delta^2}} \quad \text{where} \quad \xi = \frac{c}{c_c} \tag{36}$$

For a system for which $\xi = c/c_c$ is a small quantity, ξ^2 could be neglected comparing to the unity. And the same thing could be said for

$$\sqrt{1+\delta^2} \cong 1$$

Then the relation between δ and c is found as in the following form

$$2\xi = \delta \tag{37}$$

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