Dış Basınca Maruz Halkaların, Kemerlerin, Kabukların ve Tüplerin Stabilitesi

Stability of Rings, Archs, Shells and Tubes Under External Pressure

Hamit ÖZTEPE [1]

Bu çalışmada stabilitenin iyi bilinen problemlerine, birleştirici ve basit bir metod sunulmuştur. Bu metodda daire eksenli eğri çubukların moment diferansiyel denklemi teşkil edildikten sonra çözümlerin peryodik özelliğinden faydalanılmaktadır.

This paper presents a simple and unifying method of solution to the well-known problems of the stability. The method, after stating the moment differential equation, makes use of the fact that its solution is periodical in the case of curved bar with a circular axis.

1. Buckling of a Curved Bar with a Circular Axis

Consider the problem of stability of a curved bar with a circular axis by a radial uniformly distributed load of intensity q (Fig. 1). Isolate an elementary segment of length $ds = \rho d\varphi$ from the buckled ring. The local radius of curvature is denoted by ρ. We shall assume that this quantity differs only slightly from the initial radius of curvature r.

There are normal forces and bending moments acting at cross section of the curved bar. The moment M is related to the change in curvature by the known expression

1) Prof. Dr. Faculty of Mechanical Engineering Istanbul Technical University

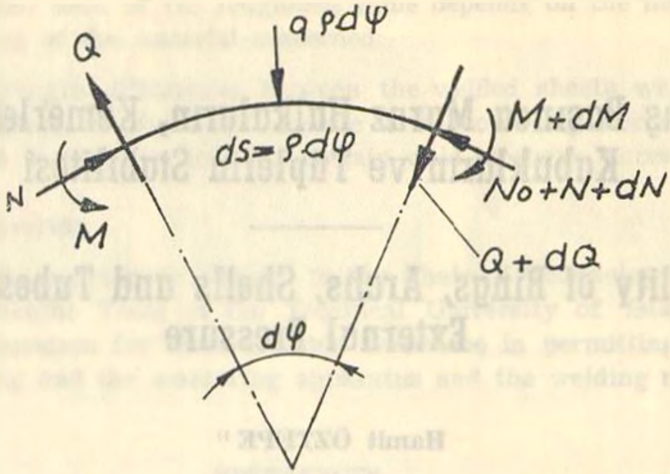

Fig. 1

$$M=EI\left(\frac{1}{\rho}-\frac{1}{r}\right) \tag{1}$$

where EI is the flexural rigidity of the curved bar. The normal force is represented as consisting of two parts: the moment N_0, the force that existed at cross sections of the curved bar before buckling, and component N representing the change in normal force due to buckling of the curved bar. Thus the normal force is equal to N_0+N. From the conditions of equilibrium in the pre-buckling state it follows that

$$N_0=qr \tag{2}$$

Let us now derive the conditions of equilibrium for the buckled element (Fig. 1). By projecting all forces on the direction of the normal, we obtain

$$q\rho\, d\varphi+dQ-(N_0+N)\, d\varphi=0 \tag{3}$$

or, taking into account the expression (2), we find

$$-q\left(\frac{1}{\rho}-\frac{1}{r}\right)+\frac{1}{\rho r}\,\frac{dQ}{d\varphi}-\frac{N}{\rho r}=0 \tag{4}$$

Substitution of expression (1) in Eq. (4) since $\rho\cong r$ gives

$$-q\frac{M}{EI}+\frac{1}{r^2}\,\frac{dQ}{d\varphi}-\frac{N}{r^2}=0 \tag{5}$$

We set up two more equations of equilibrium (Fig. 1)

$$Q\,d\varphi + dN = 0$$

or

$$\frac{dN}{d\varphi} + Q = 0 \tag{6}$$

and

$$dM + Q\,r\,d\varphi = 0$$

or

$$\frac{dM}{d\varphi} + Q\,r = 0 \tag{7}$$

We eliminate the quantities Q and N from the three equations (5), (6) and (7), to obtain

$$\frac{d^3M}{d\varphi^3} + \left(\frac{qr^3}{EI} + 1\right)\frac{dM}{d\varphi} = 0 \tag{8}$$

Using the notation

$$k^2 = \frac{qr^3}{EI} + 1 \tag{9}$$

Eq. (8) becomes

$$\frac{d^3M}{d\varphi^3} + k^2\frac{dM}{d\varphi} = 0 \tag{10}$$

2. Buckling of Circular Rings under Uniform External Pressure

Consider the problem of stability of a ring compressed by a radial uniformly distributed load of intensity q (Fig. 2). At a certain value of this load the circular form of the ring becomes unstable and the ring buckles assuming an approximately elliptical shape.

The moment $M(\varphi)$ for a closed ring is periodical and the period is 2π. The critical load (q_{cr}) for a closed ring can best be determined from the condition of periodicity of solution.

$$M(\varphi) = e^{in\varphi} \quad \text{or} \quad M = e^{-in\varphi} \tag{11}$$

We solve Eq. (10) using Eq. (11) to find

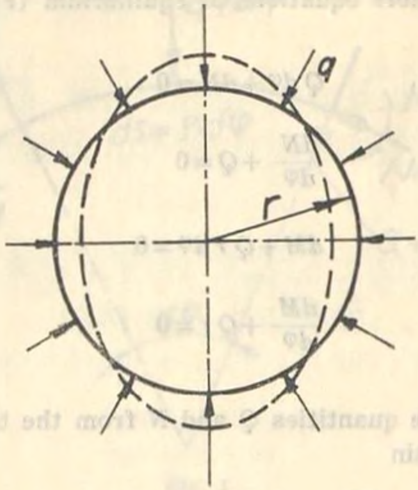

Fig. 2

$$\frac{d^3M}{d\varphi^3}+k^2\frac{dM}{d\varphi}=0 \qquad (10)$$

and

$$in^3-in\,k^2=0$$

or

$$n^2=k^2 \qquad (12)$$

We obtain, from (9) and (13)

$$q_{cr}=\frac{(n^2-1)\,EI}{r^3} \qquad (13)$$

The smallest non-zero value of q_{cr} occurs when $n=2$

$$q_{cr}=3\,EI/r^3 \qquad (14)$$

3. Buckling of a Uniformly Compressed Circular Arch

a) Curved Bar with Hinged Ends

If a curved bar with hinged ends and with its center line in the form of an arc of a circle is submitted to the action of a uniformly distributed pressure q, it will buckle as shown by the dotted line in (Fig. 3).

The solution of Eq. (10) for this case is given. In the same manner as in the previous article.

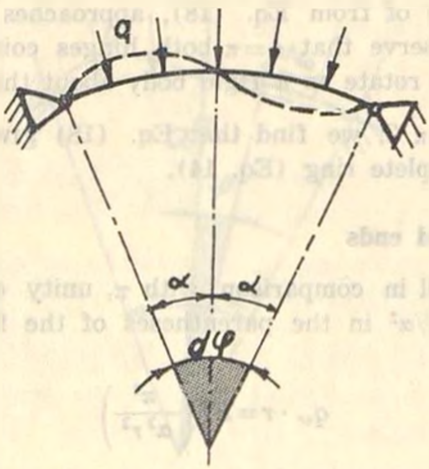

Fig. 3

$$\frac{d^3M}{d\varphi^3}+k^2\frac{dM}{d\varphi}=0 \tag{10}$$

Using the notation

$$k^2=\frac{qr^3}{EI}+1 \tag{9}$$

Taking the moment as (similar to 11)

$$M=e^{-in\varphi} \qquad n=j\frac{\pi}{\alpha} \qquad \begin{matrix} j=1,2,3\ldots \\ \varphi=2\alpha \end{matrix} \tag{15}$$

Substituting in Eq. (10), we obtain

$$i\left(j\frac{\pi}{\alpha}\right)^3-ik^2\left(j\frac{\pi}{\alpha}\right)=0$$

or

$$j^2\frac{\pi^2}{\alpha^2}=k^2 \tag{16}$$

Thus, from expression (9) we find

$$q_{cr}=\frac{EI}{r^3}\left(j^2\frac{\pi^2}{\alpha^2}-1\right) \tag{17}$$

The smallest value of q_{cr} occurs when $j=1$

$$q_{cr}=\frac{EI}{r^3}\left(\frac{\pi^2}{\alpha^2}-1\right) \qquad (18$$

When approaches the value i, e., when the arc approches the complete ring, the value of from Eq. (18), approaches zero. This can be explained, if we observe that $\alpha=\pi$ both hinges coincide and that the ring will be free to rotate as a rigid body about this common hinge.

By taking $\alpha=\pi/2$, we find that Eq. (18) gives the same value of q_{cr} as for a complete ring (Eq. 14).

Bar with hinged ends

When α is small in comparison with π, unity can be neglected in comparison with π^2/α^2 in the parentheses of the formula (18) which becomes

$$q_{cr}\cdot r = EI\left(\frac{\pi^2}{\alpha^2 r^2}\right)$$

Then the critical compressive force $p_{cr}=q_{cr}\,.\,r$ becomes equal to the critical load for a bar with hinged ends

$$P_{cr}=q_{cr}\cdot r=\frac{EI\,\pi^2}{l^2} \tag{19}$$

b) Cylindrical shell with hinged ends

Substituting $E/(1-\mu^2)$ instead of E and $h^3/12$ instead of I in Eq. (17), we obtain the equation

$$q_{cr}=\frac{Eh^3}{12\,(1-\mu^2)\,r^3}\left(\frac{\pi^2}{\alpha^2}-1\right) \tag{20}$$

which can be used in calculating the critical load for a cylindrical shell with hinged ends.

c) Curved Bar with Built Ends

If the ends of a uniformly compressed arch are built in (Fig. 4), the shape of buckling will be as shown by the dotted line.

At the middle point there will act after buckling not only the horizontal compressive force $N_0=qr$ but also a vertical shearing force Q. Designating with v the radial displacement toward the center, the bending moment at any cross section, defined by the angle φ, is

$$M(\varphi)=(qr)\,v-Qr\sin\varphi \tag{21}$$

and the differential equation (10) becomes

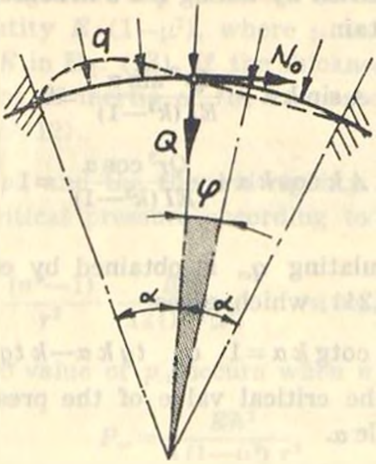

Fig. 4

$$M'' + k^2 M' = 0$$

using notation

$$k^2 = \frac{qr^3}{EI} + 1$$

From Eq. (21) and (10), we find

$$\frac{d^3V}{d\varphi^3} + k^2 \frac{dV}{d\varphi} = \frac{Qr^3}{EI} \cos\varphi$$

or

$$V'' + k^2 V = \frac{Qr^3}{EI} \sin\varphi \tag{22}$$

The general solution of this equation is

$$V = A \sin k\varphi + B \cos k\varphi + \frac{Qr^3 \sin\varphi}{EI\,(k^2-1)} \tag{23}$$

The conditions for determining the constants A and B and the shearing force Q are

$$V = \frac{d^2V}{d\varphi^2} = 0 \quad \text{at} \quad \varphi = 0 \tag{I}$$

$$V = \frac{dV}{d\varphi} = 0 \quad \text{at} \quad \varphi = \alpha \tag{II}$$

Contiditions (I) are satified by taking $B=0$ in solution (23). From conditions (II) we then obtain

$$\begin{aligned} A\sin k\alpha + \frac{Qr^3\sin\alpha}{EI(k^2-1)} &= 0 \\ A k\cos k\alpha + \frac{Qr^3\cos\alpha}{EI(k^2-1)} &= 1 \end{aligned} \tag{24}$$

The equation for calculating q_{cr} is obtained by equating to zero the determinant of Eqs. (24), which gives

$$k\,tg\,\alpha\ \mathrm{cotg}\,k\alpha = 1 \quad \text{or} \quad tg\,k\alpha - k\,tg\,\alpha = 0 \tag{25}$$

The value of k and the critical value of the presure q depend on the magnitude of the angle α.

When k is substituted in Eq. (13), the critical value of the uniform pressure is found to be

$$q_{cr} = \frac{EI}{r^3}(k^2-1) \tag{26}$$

This value of q_{cr} is always greater than that obtained from Eq. (18).

4. Tubes under External Pressure

The results obtained for a ring can readily be extended to long tubes subjected to external pressure (Fig. 5).

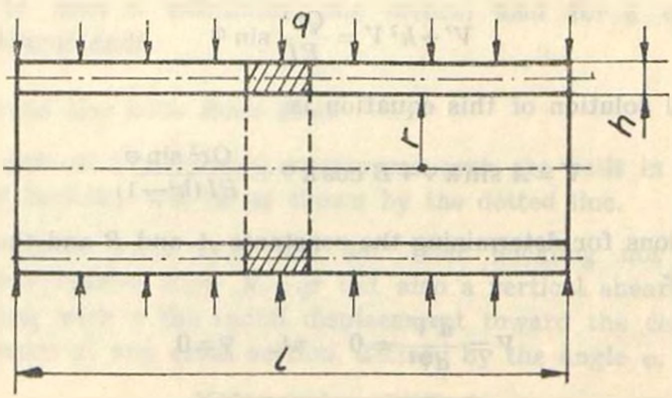

Fig. 5

An equation analogous to Eq. (10), can be obtained along a circular tube. Thus the quantity $E/(1-\mu^2)$, where μ is Poisson's ratio, should be used in place of E in Eq. (13). If the thickness of the tube is denoted by h, the moment of inertia of the cross-sectional area of the elemental ring is $I = l(h^3/12)$.

In this case $q = pl$, and the flexural rigidity of the shell is $Elh^3/12(1-\mu^2)$. Thus the critical pressure according to Eq. (13), is

$$P_{cr} = \frac{(n^2-1)}{r^3} \; \frac{Eh^3}{12(1-\mu^2)} \qquad n = 1, 2, 3 \ldots \tag{27}$$

The smallest non-zero value of p_{cr} occurs when $n = 2$

$$P_{cr} = \frac{Eh^3}{4(1-\mu^2)\,r^3} \tag{28}$$

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