DOI: 10.2478/auom-2021-0011
An. Şt. Univ. Ovidius Constanţa **Vol. 29(1),2021, 165–182**

Coincidence and Common Fixed Point Theorems via $\mathcal{C}$-Class Functions in Elliptic Valued Metric Spaces

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Abstract

The main goal of this study is to define a new metric space which is a generalization of complex valued metric spaces introduced by Azam et al. [1] using the set of elliptic numbers

$$\mathbb{E}_p = \left\{ \epsilon = \nu + i\omega : \ \nu, \omega \in \mathbb{R}, \ i^2 = p < 0 \right\},$$

and this space is named as an elliptic valued metric space. Some topological properties of this new space are examined. Also, some fixed point results are established in the setting of elliptic valued metric spaces by introducing new classes of mappings which the obtained results are real generalizations of the consequences of several fixed point theorems in the existing literature.

1 Introduction

In 1906, M. R. Fréchet realized the axiomatic development of metric spaces. Inspired by the natural development of the metric concept and its application areas in mathematical analysis many researchers have recently made various attempts to extend and generalize this concept. Some of these generalizations can be sampled as quasi metric, semimetric, 2-metric, rectangular (Brianciari

Key Words: Elliptic valued metric space, Common fixed point, $\mathcal{C}$-class Functions, Weakly Compatible Mappings.
2010 Mathematics Subject Classification: Primary 47H10; Secondary 54H25.
Received: 25.06.2020
Accepted: 01.08.2020

or generalized) metric, G-metric, partial metric, b-metric, modular metric, cone metric and complex valued metric etc.

In 2011, Azam et al.[1] and later Rouzkard et al. [2] studied on complex valued metric spaces wherein some fixed point theorems for different type mappings especially involving rational expressions were established. Indeed, complex valued metric spaces constitute a special class of cone metric spaces which defined by Huang et al. [4]. However some fixed point theorems were not established in cone metric spaces for the mappings that provide contractive conditions with rational inequalities and involving product, since the definition of a cone metric space is grounded on the underlying Banach space which is not a division ring. Therefore, some results of mathematical analysis involving divisions and products can be studied in complex valued metric spaces.

The complex number system was first studied in the 16^{th} century by Italian mathematicians G. Cardan and R. Bombelli [3], defined the complex unit as $i^2 = -1$. Since then, various mathematicians have modified this unit. The hyperbolic number system, which provides many convenience in the solution of mechanical problems, has been defined by English geometer W. Clifford [14] with $i^2 = 1, (i \neq \mp 1)$. Another modification for i is $i^2 = 0, (i \neq 0)$. The German geometer E. Study [15] introduced the dual numbers by using $i^2 = 0$ which this concept has applications in many fields such as kinematics, robotic control, spatial mechanics, and virtual reality. In the later years, these three number systems have been extended using the unit i as $p \in \mathbb{R}, i^2 = p$. The number system defined by the $i^2 = p$ is called generalized complex number system. This system is named in different ways according to the values of p. If $p < 0$ then this system is called elliptical number system (especially, if $p = -1$ then the complex number system is obtained), if $p = 0$ then this system is named parabolic or dual number system and finally if $p > 0$ then the obtained number system is called hyperbolic number system [3].

Fixed point theory has numerous applications in almost all areas of mathematical sciences. The fixed point theorem, generally known as the Banach contraction principle or Banach's fixed point theorem appeared in explicit form in Banach's thesis in 1922, which states that every contraction mapping defined on a complete metric space has a unique fixed point. Banach's contraction principle ensures, under appropriate conditions, the existence and uniqueness of a fixed point. The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last three decades so several interesting results have been found by various authors.

This paper aims to introduce the concept of elliptic valued metric spaces and examine some topological properties of this space. Bearing in mind the concept of $\mathcal{C}$-class functions suggested by Ansari in 2014 [6] we extend some contractive conditions involving rational expressions for four self mappings with weakly compatible property and obtain the coincidence and common fixed points of four self mappings in the setting of elliptic valued metric spaces.

2 Algebraic Properties of Elliptic Numbers

In the sequel, we recall some notations and definitions that will be made use of in our subsequent discussions. For more details one can see [3].

Let $\mathbb{E}_p$ denote the system of numbers

$$\mathbb{E}_p = \left\{\epsilon = \nu + i\omega : \ \nu, \omega \in \mathbb{R}, \ i^2 = p < 0\right\}.$$

For an elliptic number $\epsilon = \nu + i\omega$, the real number ν is called the real part of ϵ and ω is called the imaginary part of ϵ.

i. Summation of elliptic numbers $\epsilon_1 = \nu_1 + i\omega_1$ and $\epsilon_2 = \nu_2 + i\omega_2$ is defined as

$$\epsilon_1 + \epsilon_2 = \nu_1 + \nu_2 + i\left(\omega_1 + \omega_2\right).$$

ii. Multiplication of an elliptic number $\epsilon_1 \in \mathbb{E}_p$ with a scalar $\lambda \in \mathbb{R}$ is defined as $\lambda\epsilon_1 = \lambda\left(\nu_1 + i\omega_1\right) = \lambda\nu_1 + i\lambda\omega_1$. In addition, elliptic multiplication of two elliptic numbers $\epsilon_1, \epsilon_2 \in \mathbb{E}_p$ is defined as

$$\epsilon_1\epsilon_2 = \left(\nu_1 + i\omega_1\right)\left(\nu_2 + i\omega_2\right) = \left(\nu_1\nu_2 + p\omega_1\omega_2\right) + i\left(\nu_1\omega_2 + \nu_2\omega_1\right).$$

iii. The conjugate of an elliptic number ϵ is denoted by $\overline{\epsilon}$ and it is

$$\overline{\epsilon} = \mathrm{Re}\left(\epsilon\right) - \mathrm{Im}\left(\epsilon\right) = \nu - i\omega.$$

iv. The definition of the norm for an elliptic number is

$$\|\epsilon\|_{\mathbb{E}} = \sqrt{\epsilon\overline{\epsilon}} = \sqrt{\nu^2 - p\omega^2}.$$

It is obvious that $\mathbb{E}_p$ is a two-dimensional vector space over field $\mathbb{R}$ with addition and scalar multiplication. Hence one to one mapping can be characterised from $\mathbb{E}_p$ to $\mathbb{R}^2$ and each elliptical number $\epsilon = \nu + i\omega$ can be expressed uniquely in the (ordinary) plane. This plane is called elliptical plane and in an elliptical plane the distance between two elliptic numbers $\epsilon_1 = (\nu_1, \omega_1)$ and $\epsilon_2 = (\nu_2, \omega_2)$ is designated by

$$\|\epsilon_1 - \epsilon_2\|_{\mathbb{E}} = \sqrt{\left(\nu_1 - \nu_2\right)^2 - p(\omega_1 - \omega_2)^2}.$$

In this elliptic plane, the set of all elliptical numbers located 1 unit away from the origin is ellipse and is expressed by the equation $\nu^2 - p\omega^2 = 1$ [3].

Throughout this study, θ is used as the zero vector of the space $\mathbb{E}_p$.
Now, define a partial order $\preceq$ on $\mathbb{E}_p$ as follows:

Let $\epsilon_1 = \nu_1 + i\omega_1 \in \mathbb{E}_p$ and $\epsilon_2 = \nu_2 + i\omega_2 \in \mathbb{E}_p$,

$$\epsilon_1 \preceq \epsilon_2 \Leftrightarrow \operatorname{Re}(\epsilon_1) \leq \operatorname{Re}(\epsilon_2) \text{ and } \operatorname{Im}(\epsilon_1) \leq \operatorname{Im}(\epsilon_2). \tag{2.1}$$

It follows that $\epsilon_1 \preceq \epsilon_2$ if any one of the following statements holds:

o_1. $\operatorname{Re}(\epsilon_1) = \operatorname{Re}(\epsilon_2)$ and $\operatorname{Im}(\epsilon_1) < \operatorname{Im}(\epsilon_2)$;

o_2. $\operatorname{Re}(\epsilon_1) < \operatorname{Re}(\epsilon_2)$ and $\operatorname{Im}(\epsilon_1) = \operatorname{Im}(\epsilon_2)$;

o_3. $\operatorname{Re}(\epsilon_1) < \operatorname{Re}(\epsilon_2)$ and $\operatorname{Im}(\epsilon_1) < \operatorname{Im}(\epsilon_2)$;

o_4. $\operatorname{Re}(\epsilon_1) = \operatorname{Re}(\epsilon_2)$ and $\operatorname{Im}(\epsilon_1) = \operatorname{Im}(\epsilon_2)$.

In particular, the expression $\epsilon_1 \precnsim \epsilon_2$ $(\epsilon_1 \neq \epsilon_2)$ will be used if one of o_1, o_2 and o_3 is provided and the expression $\epsilon_1 \prec \epsilon_2$ will only be used if o_3 is provided. Some basic features of the partial order $\preceq$ on $\mathbb{E}_p$ can be given as follows:

po_1. If $\theta \preceq \epsilon_1 \precnsim \epsilon_2$, then $\|\epsilon_1\|_{\mathbb{E}} < \|\epsilon_2\|_{\mathbb{E}}$.

po_2. $\epsilon_1 \preceq \epsilon_2$ is equivalent to $\epsilon_1 - \epsilon_1 \preceq \theta$.

po_3. If $\epsilon_1 \preceq \epsilon_2$ and $\epsilon_2 \preceq \epsilon_3$, then $\epsilon_1 \preceq \epsilon_3$.

po_4. If $\epsilon_1 \preceq \epsilon_2$ and $\Lambda > 0, (\Lambda \in \mathbb{R})$, then $\Lambda\epsilon_1 \preceq \Lambda\epsilon_2$.

po_5. $\theta \preceq \epsilon_1$ and $\theta \preceq \epsilon_2$ do not imply $\theta \preceq \epsilon_1\epsilon_2$.

3 Elliptic Valued Metric Spaces and Some Topological Properties

Inspired by the method Azam et al. used in their study [1] we introduce elliptic valued metric space and examine some topological properties which are necessary for our main discussion.

Definition 3.1. Let Ξ be a non empty set. A function $\mathbf{e} : \Xi \times \Xi \to \mathbb{E}_p$ is an elliptic valued metric on Ξ if it satisfies the following properties:

$(\mathbb{E}_1)$ $\theta \preceq \mathbf{e}(\sigma, \varsigma)$ for all $\sigma, \varsigma \in \Xi$;

($\mathbb{E}_2$) $\mathbf{e}(\sigma,\varsigma)=\theta$ if and only if $\sigma=\varsigma$;

($\mathbb{E}_3$) $\mathbf{e}(\sigma,\varsigma)=\mathbf{e}(\varsigma,\sigma)$ for all $\sigma,\varsigma\in\Xi$;

($\mathbb{E}_4$) $\mathbf{e}(\sigma,\varsigma)\preceq\mathbf{e}(\sigma,\tau)+\mathbf{e}(\tau,\varsigma)$ for all $\sigma,\varsigma,\tau\in\Xi$.

In this case, the pair $(\Xi,\mathbf{e})$ is called an elliptic valued metric space.

Example 3.1. Let $\Xi=\mathbb{E}_p$ be the set of elliptic numbers. Define $\mathbf{e}:\mathbb{E}_p\times\mathbb{E}_p\to\mathbb{E}_p$ by

$$\mathbf{e}\left(\epsilon_1,\epsilon_2\right)=\left\|\nu_1-\nu_2\right\|_{\mathbb{E}}+i\left\|\omega_1-\omega_2\right\|_{\mathbb{E}},$$

where $\epsilon_1=\nu_1+i\omega_1,\ \epsilon_2=\nu_2+i\omega_2\in\mathbb{E}_p$. Then $(\mathbb{E}_p,\mathbf{e})$ is an elliptic valued metric space.

Example 3.2. Let $\Xi=\mathbb{E}_p$ be the set of elliptic numbers and $\epsilon_1,\epsilon_2\in\Xi$. Define the mapping $\mathbf{e}:\mathbb{E}_p\times\mathbb{E}_p\to\mathbb{E}_p$ by

$$\mathbf{e}\left(\epsilon_1,\epsilon_2\right)=\left\|\epsilon_1-\epsilon_2\right\|_{\mathbb{E}}e^{i\Theta_p},\Theta_p\in\left[0,\frac{\pi\left(p-1\right)}{8p}\right],$$

where Θ_p is argument of ϵ_1 and ϵ_2 and $p<0$ and $p\in\mathbb{R}$. Then one can easily check that $(\mathbb{E}_p,\mathbf{e})$ is an elliptic valued metric space.

3.1 On Some Topology Related to Elliptic Valued Metric Spaces

In this subsection, some topological properties related to elliptic valued metric space will be mentioned.

Definition 3.2. Let $(\Xi,\mathbf{e})$ be an elliptic valued metric space. A point $\sigma\in\Xi$ is called $\mathbf{e}$-interior point of a set $\mathbb{A}\subseteq\Xi$ whenever there exists $\theta\prec\delta\in\mathbb{E}_p$ such that

$$\mathbf{B}_{\mathbb{E}}(\sigma,\delta)=\{\varsigma\in\Xi:\mathbf{e}(\sigma,\varsigma)\prec\delta\}\subseteq\mathbb{A},$$

where $\mathbf{B}_{\mathbb{E}}(\sigma,\delta)$ is an $\mathbf{e}$-open ball. Then $\mathbf{B}_{\mathbb{E}}[\sigma,\delta]=\{\varsigma\in\Xi:\mathbf{e}(\sigma,\varsigma)\preceq\delta\}$ is an $\mathbf{e}$-closed ball in the setting of elliptic valued metric space.

Definition 3.3. Let $(\Xi,\mathbf{e})$ be an elliptic valued metric space. A point $\sigma\in\Xi$ is called $\mathbf{e}$-limit point of a set $\mathbb{A}\subseteq\Xi$ whenever for every $\theta\prec\delta\in\mathbb{E}_p$,

$$(\mathbf{B}_{\mathbb{E}}(\sigma,\delta)-\{\sigma\})\cap\mathbb{A}\neq\emptyset.$$

$\mathbb{A}$ is called $\mathbf{e}$-open whenever each element of $\mathbb{A}$ is an $\mathbf{e}$-interior point of $\mathbb{A}$. Moreover, a subset $\mathbb{F}\subseteq\Xi$ is called $\mathbf{e}$-closed whenever each $\mathbf{e}$-limit point of $\mathbb{F}$ belongs to $\mathbb{F}$. The family

$$\Sigma=\{\mathbf{B}_{\mathbb{E}}(\sigma,\delta):\sigma\in\Xi,\theta\prec\delta\},$$

is a sub-basis for a Hausdorff topology on Ξ.

Definition 3.4. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be a sequence in $(\Xi, \mathbf{e})$. The sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ is said to $\mathbf{e}$-converge $\sigma \in \Xi$ if and only if $\theta \prec \delta \in \Xi$, there exists $n_0 \in \mathbb{N}$ such that $\mathbf{e}(\sigma_n, \sigma) \prec \delta$ for all $n > n_0$. It is denoted by $\sigma_n \to \sigma$ as $n \to \infty$ or $\lim\limits_{n\to\infty} \sigma_n = \sigma$.

Definition 3.5. A sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ in an elliptic valued metric space $(\Xi, \mathbf{e})$ is said to be an $\mathbf{e}$-Cauchy sequence if and only if for any $\theta \prec \delta \in \Xi$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\mathbf{e}(\sigma_n, \sigma_{n+m}) \prec \delta$, where $m \in \mathbb{N}$.

An elliptic valued metric space $(\Xi, \mathbf{e})$ is said to be $\mathbf{e}$-complete if and only if every $\mathbf{e}$-Cauchy sequence in Ξ $\mathbf{e}$-converges in Ξ.

We require the following lemmas.

Lemma 3.1. *Let $(\Xi, \mathbf{e})$ be an elliptic valued metric space and $\{\sigma_n\}_{n\in\mathbb{N}}$ be a sequence in Ξ. Then the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ $\mathbf{e}$-converges to $\sigma \in \Xi$ if and only if $\|\mathbf{e}(\sigma_n, \sigma)\|_{\mathbb{E}} \to 0$ as $n \to \infty$.*

Proof. Let us assume that that $\{\sigma_n\}$ $\mathbf{e}$-converges to σ. Choose

$$\delta = \frac{\varepsilon}{\sqrt{1-p}} + i\frac{\varepsilon}{\sqrt{1-p}},$$

for a given positive real number ε and $p < 0$. Then $\theta \prec \delta \in \mathbb{E}_p$ and there is a natural number n_0 such that $\mathbf{e}(\sigma_n, \sigma) \prec \delta$ for all $n > n_0$ and $p < 0$. Thus, $\|\mathbf{e}(\sigma_n, \sigma)\|_{\mathbb{E}} < \|\delta\|_{\mathbb{E}} = \varepsilon$ for all $n > n_0$. Consequently, $\lim\limits_{n\to\infty} \|\mathbf{e}(\sigma_n, \sigma)\|_{\mathbb{E}} = 0$. Conversely, granted that $\lim\limits_{n\to\infty} \|\mathbf{e}(\sigma_n, \sigma)\|_{\mathbb{E}} = 0$. Then for $\theta \prec \delta \in \mathbb{E}_p$, there exists a real number $\zeta > 0$, such that for $\epsilon \in \mathbb{E}_p$ and $\|\epsilon\|_{\mathbb{E}} < \zeta \Rightarrow \epsilon \prec \delta$. For this ζ, there exists a natural number n_0 such that $\|\mathbf{e}(\sigma_n, \sigma)\|_{\mathbb{E}} < \zeta$ for all $n > n_0$. This concludes that $\mathbf{e}(\sigma_n, \sigma) \prec \delta$ for all $n > n_0$. Hence $\{\sigma_n\}_{n\in\mathbb{N}}$ $\mathbf{e}$-converges to σ. □

Lemma 3.2. *Let $(\Xi, \mathbf{e})$ be an elliptic valued metric space and $\{\sigma_n\}_{n\in\mathbb{N}}$ be a sequence in Ξ. Then the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $\mathbf{e}$-Cauchy sequence if and only if $\|\mathbf{e}(\sigma_n, \sigma_{n+k})\|_{\mathbb{E}} \to 0$ as $n \to \infty$, where $k \in \mathbb{N}$.*

Proof. Suppose that $\{\sigma_n\}_{n\in\mathbb{N}}$ is a $\mathbf{e}$-Cauchy sequence. Choose

$$\delta = \frac{\varepsilon}{\sqrt{1-p}} + i\frac{\varepsilon}{\sqrt{1-p}},$$

for a given positive real number ε and $p < 0$. Then $\theta \prec \delta \in \mathbb{E}_p$ and there exists a natural number n_0 such that $\mathbf{e}(\sigma_n, \sigma_{n+k}) \prec \delta$ for all $n > n_0$, $p < 0$

and $k \in \mathbb{N}$. Therefore $\|\mathbf{e}(\sigma_n, \sigma_{n+k})\|_{\mathbb{E}} < \|\delta\| = \varepsilon$ for all $n_0 \in \mathbb{N}$ and $k \in \mathbb{N}$. Then the following statement is obtained;

$$\lim_{n\to\infty} \|\mathbf{e}(\sigma_n, \sigma_{n+k})\|_{\mathbb{E}} = 0,$$

where $k \in \mathbb{N}$.
Conversely, suppose that $\|\mathbf{e}(\sigma_n, \sigma_{n+k})\|_{\mathbb{E}} \to 0$ as $n \to \infty$ and $k \in \mathbb{N}$. Then given $\theta \prec \delta \in \mathbb{E}_p$, there is a real number $\zeta > 0$, such that for $\epsilon \in \mathbb{E}_p$ and $\|\epsilon\|_{\mathbb{E}} < \zeta \Rightarrow \epsilon \prec \delta$. For this ζ, there exist a natural number n_0 such that $\|e(\sigma_n, \sigma_{n+k})\|_{\mathbb{E}} < \zeta$ for all $n > n_0$ and $k \in \mathbb{N}$, which means $\mathbf{e}(\sigma_n, \sigma_{n+k}) \prec \delta$ for all $n > n_0$ and $k \in \mathbb{N}$, so $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $\mathbf{e}$-Cauchy sequence. □

We use the notations $\mathbb{K}$ and Int$\mathbb{K}$ to indicate the following subsets of $\mathbb{E}_p$.

$$\mathbb{K} = \{\epsilon \in \mathbb{E}_p : \theta \preceq \epsilon\} = \{\epsilon = \nu + i\omega \in \mathbb{E}_p : \nu \geq 0, \omega \geq 0\},$$

and

$$\text{Int}\mathbb{K} = \{\epsilon \in \mathbb{E}_p : \theta \prec \epsilon\} = \{\epsilon = \nu + i\omega \in \mathbb{E}_p : \nu > 0, \omega > 0\}.$$

In the set $\mathbb{K}$ every increasing sequence which is bounded from above is **e**-convergent (or every decreasing sequence which is bounded from below is **e**-convergent).

Remark 3.1. In an elliptic valued metric space $(\Xi, \mathbf{e})$ the following statements hold true. Let $\{\sigma_n\}_{n\in\mathbb{N}}, \{\varsigma_n\}_{n\in\mathbb{N}}$ be sequences in $(\Xi, \mathbf{e})$.

i. If $\theta \preceq \sigma_n \preceq \varsigma_n$, for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \sigma_n = \sigma$ and $\lim_{n\to\infty} \varsigma_n = \varsigma \Rightarrow \theta \preceq \sigma \preceq \varsigma$.

ii. If $\sigma_n \preceq \varsigma_n \preceq \tau_n$, for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \tau_n = \sigma \Rightarrow \lim_{n\to\infty} \varsigma_n = \sigma.$$

iii. The **e**-limit of a **e**-convergent sequence in an elliptic valued metric space is unique.

Definition 3.6. Let $\hbar : \mathbb{K} \to \mathbb{K}$ be a function.

i. $\hbar$ is monotone increasing if for any $\sigma, \varsigma \in \mathbb{K}, \sigma \preceq \varsigma \Leftrightarrow \hbar(\sigma) \preceq \hbar(\varsigma)$.

ii. $\hbar$ is said to be **e**-continuous at $\sigma_0 \in \mathbb{K}$ if for any sequence $\{\sigma_n\}_{n\in\mathbb{N}} \in \mathbb{K}$,

$$\sigma_n \to \sigma_0 \Rightarrow \hbar(\sigma_n) \to \hbar(\sigma_0).$$

Recently, Ansari et al.[5] introduced complex $\mathcal{C}$-class functions. Inspiring the idea of them we redefine $\mathcal{C}$-class functions in the setting of elliptic valued metric space.

Definition 3.7. Let $\mathfrak{F} : \mathbb{K} \times \mathbb{K} \to \mathbb{E}_p$ be a function. For any $\sigma, \varsigma \in \mathbb{K}$ the following conditions hold:

$\mathcal{C}_1$. $\mathfrak{F}$ is $\mathbf{e}$-continuous;

$\mathcal{C}_2$. $\mathfrak{F}(\sigma, \varsigma) \preceq \sigma$;

$\mathcal{C}_3$. $\mathfrak{F}(\sigma, \varsigma) = \sigma$ implies that either $\sigma = \theta$ or $\varsigma = \theta$;

$\mathcal{C}_4$. $\mathfrak{F}(\theta, \theta) = \theta$.

Then the function $\mathfrak{F}$ is called an elliptic valued $\mathcal{C}$-class function.

The following functions can be given as an example of elliptic valued $\mathcal{C}$-class functions.

Example 3.3. Let $\sigma, \varsigma \in \mathbb{K}$.

i. $\mathfrak{F}(\sigma, \varsigma) = \sigma - \varsigma$;

ii. for some $\eta \in (0, 1)$, $\mathfrak{F}(\sigma, \varsigma) = \eta\sigma$;

iii. $\mathfrak{F}(\sigma, \varsigma) = \sigma - \chi(\sigma)$, where $\chi : \mathbb{K} \to \mathbb{K}$ is $\mathbf{e}$-continuous, $\chi(\theta) = \theta$ and $\chi(\sigma) \succ \theta$ if $\sigma \succ \theta$;

iv. $\mathfrak{F}(\sigma, \varsigma) = \sigma\Gamma(\sigma)$, where $\Gamma : \mathbb{K} \to [0, 1)$ $\mathbf{e}$-continuous.

In our following discussion we need to redefine the following control functions. We denote by Υ the set of all functions $\kappa : \text{Int}\mathbb{K} \cup \{\theta\} \to \text{Int}\mathbb{K} \cup \{\theta\}$ satisfying

i. κ is $\mathbf{e}$-continuous and non-decreasing,

ii. $\kappa(\sigma) \succ \theta$ if $\sigma \succ \theta$ and $\kappa(\theta) = \theta$.

The following lemma is necessary for proofs of our results. Its proof runs in the same line with the proof of lemma given by Choudhury et al. [7], so we omit them.

Lemma 3.3. *Let $(\Xi, \mathbf{e})$ be an elliptic valued metric space such that $\mathbf{e}(\sigma, \varsigma) \in$ Int$\mathbb{K}$, for $\sigma, \varsigma \in \Xi$ with $\sigma \neq \varsigma$. Let $\chi \in \Upsilon$ be such that either $\chi(\sigma) \preceq \mathbf{e}(\sigma, \varsigma)$ or $\mathbf{e}(\sigma, \varsigma) \preceq \chi(\sigma)$, for $\sigma \in$ Int$\mathbb{K}$ and $\sigma, \varsigma \in \Xi$. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be a sequence in Ξ for which $\{\mathbf{e}(\sigma_n, \varsigma_n)\}_{n\in\mathbb{N}}$ is monotonic decreasing. Then $\{\mathbf{e}(\sigma_n, \varsigma_n)\}_{n\in\mathbb{N}}$ is $\mathbf{e}$-convergent to either $\delta = \theta$ or $\delta \in$ Int$\mathbb{K}$.*

4 Fixed Point Theorems for Mappings via Elliptic Valued $\mathcal{C}$-Class Functions in Elliptic Valued Metric Spaces

In the following, we need the concept of weakly compatibility to prove our theorems which were introduced by Jungck [8].

Definition 4.1. Let $\hbar$ and $\wp$ be self maps of a set Ξ. If $\omega = \hbar\nu = \wp\nu$ for some $\nu \in \Xi$, then ν is called a coincidence point of $\hbar$ and $\wp$, and ω is called a point of coincidence of $\hbar$ and $\wp$. Self maps $\hbar$ and $\wp$ are said to be weakly compatible if they commute at their coincidence point; that is; if $\hbar\nu = \wp\nu$ for some $\nu \in \Xi$, then $\hbar\wp\nu = \wp\hbar\nu$.

Now we present our main results by expanding the contractive conditions of [9] and also in its references [10], [11] via elliptic valued $\mathcal{C}$-class function in the setting of elliptic valued metric space.

Definition 4.2. Let $(\Xi, \mathbf{e})$ be an elliptic valued metric space, $M, N, P, R : \Xi \to \Xi$ be self mappings. Suppose there exist $\kappa, \chi \in \Upsilon$ and $\mathfrak{F} \in \mathcal{C}$ such that for all $\sigma, \varsigma \in \Xi$ with

$$\kappa\left(\mathbf{e}\left(M\sigma, N\varsigma\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\sigma,\varsigma\right)\right), \chi\left(\mathrm{Z}\left(\sigma,\varsigma\right)\right)\right), \tag{4.1}$$

where

$$\begin{cases} \mathrm{Z}\left(\sigma,\varsigma\right) = \frac{\mathbf{e}(P\sigma,M\sigma)\mathbf{e}(P\sigma,N\varsigma)}{\mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,R\varsigma)+\mathbf{e}(P\sigma,N\varsigma)} \\ \text{for all } \sigma,\varsigma \in \Xi \text{ with } \sigma \neq \varsigma \text{ if } \mathbf{e}\left(P\sigma, M\sigma\right) + \mathbf{e}\left(P\sigma, R\varsigma\right) + \mathbf{e}\left(P\sigma, N\varsigma\right) \neq \theta; \\ \mathbf{e}\left(M\sigma, N\varsigma\right) > \theta, \quad \text{if } \mathbf{e}\left(P\sigma, M\sigma\right) + \mathbf{e}\left(P\sigma, R\varsigma\right) + \mathbf{e}\left(P\sigma, N\varsigma\right) = \theta. \end{cases}$$

Then the pair $\{M, N\}$ is called a generalized $\mathcal{C}_{\mathbb{E}}$-$\{P, R\}$ contractive pair of type I.

Theorem 4.1. *Let $(\Xi, \mathbf{e})$ be an $\mathbf{e}$-complete elliptic valued metric space and $M, N, P, R : \Xi \to \Xi$ be self mappings. Assume that the following statements hold:*

i. the pair $\{M, N\}$ is generalized $\mathcal{C}_{\mathbb{E}}$-$\{P, R\}$ contractive pair of type I;

ii. $N\Xi \subseteq P\Xi$ and $M\Xi \subseteq R\Xi$;

iii. one of the ranges $M\Xi, N\Xi, P\Xi$ and $R\Xi$ is $\mathbf{e}$-closed.

Then $C(M,P) = \{x \in \Xi : x = Mz = Pz\} \neq \emptyset$ *and* $C(N,R) = \{x \in \Xi : x = Nz = Rz\} \neq \emptyset$. *In addition, if* $\{M,P\}$ *and* $\{N,R\}$ *are weakly compatible pairs then the mappings* M, N, P *and* R *have a unique common fixed point in* Ξ.

Proof. Without loss of generality assume that

$$\mathbf{e}(P\sigma, M\sigma) + \mathbf{e}(P\sigma, R\varsigma) + \mathbf{e}(P\sigma, N\varsigma) \neq \theta$$

then $\mathbf{e}(M\sigma, N\varsigma) > \theta_{\mathbb{E}}$. For an arbitrary point $\sigma_0 \in \Xi$, by the hypothesis of theorem condition $i.$, choose a point $\sigma_1 \in \Xi$ such that $M\sigma_0 = R\sigma_1 = \varsigma_0$. For a point σ_1, there exists a point σ_2 in Ξ such that $N\sigma_1 = P\sigma_2 = \varsigma_1$. Inductively, we can construct a sequence $\{\varsigma_n\}$ in Ξ such that

$$M\sigma_{2n} = R\sigma_{2n+1} = \varsigma_{2n}$$

$$N\sigma_{2n+1} = P\sigma_{2n+2} = \varsigma_{2n+1}, \tag{4.2}$$

for all $n = 0, 1, 2, ...$ Let us suppose that $\varsigma_{2n+1} \neq \varsigma_{2n}$ for all $n = 0, 1, 2, ...$ First, we show that $\lim_{n\to\infty} \mathbf{e}(\varsigma_n, \varsigma_{n+1}) = \theta$. Using the facts (4.1) and (4.2), we get

$$\kappa(\mathbf{e}(\varsigma_{2n}, \varsigma_{2n+1})) = \kappa(\mathbf{e}(M\sigma_{2n}, N\sigma_{2n+1})) \preceq \mathfrak{F}(\kappa(\mathrm{Z}(\sigma_{2n}, \sigma_{2n+1})), \chi(\mathrm{Z}(\sigma_{2\mathrm{n}}, \sigma_{2\mathrm{n}+1}))),$$

where

$$\mathrm{Z}(\sigma_{2\mathrm{n}}, \sigma_{2\mathrm{n}+1}) = \frac{\mathbf{e}(\mathrm{P}\sigma_{2\mathrm{n}}, \mathrm{M}\sigma_{2\mathrm{n}})\mathbf{e}(\mathrm{P}\sigma_{2\mathrm{n}}, \mathrm{N}\sigma_{2\mathrm{n}+1})}{\mathbf{e}(\mathrm{P}\sigma_{2\mathrm{n}}, \mathrm{M}\sigma_{2\mathrm{n}}) + \mathbf{e}(\mathrm{P}\sigma_{2\mathrm{n}}, \mathrm{R}\sigma_{2\mathrm{n}+1}) + \mathbf{e}(\mathrm{P}\sigma_{2\mathrm{n}}, \mathrm{N}\sigma_{2\mathrm{n}+1})}$$

$$= \frac{\mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n})\mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n+1})}{\mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n}) + \mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n}) + \mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n+1})}$$

$$< \frac{\mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n})\mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n+1})}{\mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n+1})} < \mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n}).$$

Since $\kappa, \chi \in \Upsilon$ is non-decresing we get

$$\kappa(\mathbf{e}(\varsigma_{2n}, \varsigma_{2n+1})) \preceq \mathfrak{F}(\kappa(\mathbf{e}(\varsigma_{2\mathrm{n}-1}, \varsigma_{2\mathrm{n}})), \chi(\mathbf{e}(\varsigma_{2\mathrm{n}-1}, \varsigma_{2\mathrm{n}}))).$$

Owing to the property of $\mathfrak{F}$

$$\begin{aligned} \kappa(\mathbf{e}(\varsigma_{2n}, \varsigma_{2n+1})) &\preceq \mathfrak{F}(\kappa(\mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n})), \chi(\mathbf{e}(\varsigma_{2\mathrm{n}-1}, \varsigma_{2\mathrm{n}}))) \\ &\preceq \kappa(\mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n})), \end{aligned} \tag{4.3}$$

and monotonicity of κ we have

$$\mathbf{e}(\varsigma_{2n}, \varsigma_{2n+1}) \prec \mathbf{e}(\varsigma_{2n-1}, \varsigma_{2n}),$$

which yields that the sequence $\{\mathbf{e}\left(\varsigma_{2n},\varsigma_{2n+1}\right)\}$ is a decreasing sequence. Hence by lemma(3.3), there exists an $\tau\in\mathbb{K}$ with either $\tau=\theta$ or $\tau\in\operatorname{Int}\mathbb{K}$ such that

$$\mathbf{e}\left(\varsigma_{2n},\varsigma_{2n+1}\right)\to\tau \quad \text{as} \quad n\to\infty. \tag{4.4}$$

Taking limit as $n\to\infty$ in (4.3), using (4.4) and $\mathbf{e}$-continuity of the functions κ,χ and $\mathfrak{F}$, we obtain

$$\kappa\left(\tau\right)\preceq\mathfrak{F}\left(\kappa\left(\tau\right),\chi\left(\tau\right)\right)\preceq\kappa\left(\tau\right),$$

and due to the property of $\mathfrak{F}$ we get

$$\mathfrak{F}\left(\kappa\left(\tau\right),\chi\left(\tau\right)\right)=\kappa\left(\tau\right),$$

and concluded that $\kappa\left(\tau\right)=\theta=\chi\left(\tau\right)\Leftrightarrow\tau=\theta$, in other words

$$\mathbf{e}\left(\varsigma_{2n},\varsigma_{2n+1}\right)\to\theta \quad \text{as} \quad n\to\infty. \tag{4.5}$$

Next, we show that $\{\varsigma_{2n}\}$ is a $\mathbf{e}$-Cauchy sequence. If $\{\varsigma_{2n}\}$ is not a $\mathbf{e}$-Cauchy sequence, then there exists $\delta\in\mathbb{E}_p$ with $\theta\prec\delta$, for all $N\in\mathbb{N}$, there exist $m,n\in\mathbb{N}$ with $n>m\geq n_0$ such that

$$\mathbf{e}(\varsigma_m,\varsigma_n)\not\prec\chi(\delta).$$

Hence by the property of χ, $\chi(\delta)\preceq\mathbf{e}(\varsigma_m,\varsigma_n)$. Therefore, there exist two sequences $\{\varsigma_{2m(k)}\}$ and $\{\varsigma_{2n(k)}\}$ of $\{\varsigma_{2n}\}$ where $m(k)$ is smallest integer such that $m(k)>n(k)\geq k$ and

$$\mathbf{e}(\varsigma_{2m(k)},\varsigma_{2n(k)})\succeq\chi\left(\delta\right), \tag{4.6}$$

and

$$\mathbf{e}(\varsigma_{2m(k)-2},\varsigma_{2n(k)})\prec\chi\left(\delta\right). \tag{4.7}$$

Now, from (4.6-4.7) we obtain

$$\lim_{k\to\infty}\mathbf{e}\left(\varsigma_{2m(k)},\varsigma_{2n(k)}\right)=\chi\left(\delta\right), \tag{4.8}$$

$$\lim_{k\to\infty}\mathbf{e}\left(\varsigma_{2m(k)+1},\varsigma_{2n(k)}\right)=\chi\left(\delta\right), \tag{4.9}$$

$$\lim_{k\to\infty}\mathbf{e}\left(\varsigma_{2m(k)+2},\varsigma_{2n(k)+1}\right)=\chi\left(\delta\right), \tag{4.10}$$

$$\lim_{k\to\infty}\mathbf{e}\left(\varsigma_{2m(k)+1},\varsigma_{2n(k)+1}\right)=\chi\left(\delta\right). \tag{4.11}$$

In inequality (4.1), let us suppose that $\sigma = \varsigma_{2m(k)+2}$ and $\varsigma = \varsigma_{2n(k)+1}$, for all $k \geq 0$. Then,

$$\begin{aligned} &\kappa\left(\mathbf{e}\left(\varsigma_{2m(k)+2}, \varsigma_{2n(k)+1}\right)\right) = \kappa\left(\mathbf{e}\left(M\sigma_{2m(k)+2}, N\sigma_{2n(k)+1}\right)\right) \\ &\preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\sigma_{2m(k)+2}, \sigma_{2n(k)+1}\right)\right), \chi\left(\mathrm{Z}\left(\sigma_{2\mathrm{m(k)}+2}, \sigma_{2\mathrm{n(k)}+1}\right)\right)\right), \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} &\mathrm{Z}\left(\sigma_{2m(k)+2}, \sigma_{2n(k)+1}\right) \\ &= \frac{\mathbf{e}\left(P\sigma_{2m(k)+2}, M\sigma_{2m(k)+2}\right)\mathbf{e}\left(P\sigma_{2m(k)+2}, N\sigma_{2n(k)+1}\right)}{\mathbf{e}\left(P\sigma_{2m(k)+2}, M\sigma_{2m(k)+2}\right)+\mathbf{e}\left(P\sigma_{2m(k)+2}, R\sigma_{2n(k)+1}\right)+\mathbf{e}\left(P\sigma_{2m(k)+2}, N\sigma_{2n(k)+1}\right)} \\ &= \frac{\mathbf{e}\left(\varsigma_{2m(k)+1}, \varsigma_{2m(k)+2}\right)\mathbf{e}\left(\varsigma_{2m(k)+1}, \varsigma_{2n(k)+1}\right)}{\mathbf{e}\left(\varsigma_{2m(k)+1}, \varsigma_{2m(k)+2}\right)+\mathbf{e}\left(\varsigma_{2m(k)+1}, \varsigma_{2n(k)}\right)+\mathbf{e}\left(\varsigma_{2m(k)+1}, \varsigma_{2n(k)+1}\right)} \end{aligned}$$

With the equalities (4.8-4.11) we obtain that

$$\lim_{k\to\infty} \mathrm{Z}\left(\sigma_{2m(k)+2}, \sigma_{2n(k)+1}\right) = \theta. \tag{4.13}$$

In inequality (4.12), letting $k \to \infty$ using (4.13) and the continuity property of the functions κ, χ and $\mathfrak{F}$, we have

$$\kappa\left(\chi\left(\delta\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\theta\right), \chi\left(\theta\right)\right) = \theta,$$

so $\chi(\delta) = \theta$ and $\delta = \theta$ implies a contradiction. Hence for all $n \in \mathbb{N}$, $\{\varsigma_n\}$ is an $\mathbf{e}$-Cauchy sequence. As $(\Xi, \mathbf{e})$ is $\mathbf{e}$-complete, then it yields that $\{\varsigma_n\}$ and hence any subsequence thereof, $\mathbf{e}$-converge to $\xi \in \Xi$. So the sequences

$$\{M\sigma_{2n}\}, \{N\sigma_{2n+1}\}, \{P\sigma_{2n+2}\}, \{R\sigma_{2n+1}\} \to \xi \in \Xi \tag{4.14}$$

as $n \to \infty$

Case i. Suppose $P\Xi$ is $\mathbf{e}$-closed. In view of (4.14), we have $\xi \in P\Xi$ such that $\xi = P\upsilon$. In (4.1), applying $\sigma = \upsilon$ and $\varsigma = \varsigma_{2n+1}$ we obtain

$$\kappa\left(\mathbf{e}\left(M\upsilon, N\varsigma_{2n+1}\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\upsilon, \varsigma_{2n+1}\right)\right), \chi\left(Z\left(\upsilon, \varsigma_{2n+1}\right)\right)\right), \tag{4.15}$$

where

$$\mathrm{Z}\left(\upsilon, \varsigma_{2n+1}\right) = \frac{\mathbf{e}\left(P\upsilon, M\upsilon\right)\mathbf{e}\left(P\upsilon, N\varsigma_{2n+1}\right)}{\mathbf{e}\left(P\upsilon, M\upsilon\right) + \mathbf{e}\left(P\upsilon, R\varsigma_{2n+1}\right) + \mathbf{e}\left(P\upsilon, N\varsigma_{2n+1}\right)}.$$

Letting $k \to \infty$ in inequality (4.15) and using the continuity of the functions κ, χ and $\mathfrak{F}$ the following statement is obtained.

$$\kappa\left(\left\|\mathbf{e}\left(M\upsilon, \xi\right)\right\|_{\mathbb{E}}\right) \preceq \mathfrak{F}\left(\kappa\left(\left\|\mathrm{Z}\left(\upsilon, \xi\right)\right\|_{\mathbb{E}}\right), \chi\left(\left\|\mathrm{Z}\left(\upsilon, \xi\right)\right\|_{\mathbb{E}}\right)\right),$$

and $M\upsilon = \xi$. Hence
$$M\upsilon = P\upsilon = \xi.$$
Because $\xi = M\upsilon \in M\Xi \subseteq R\Xi$, we have $\xi \in R\Xi$, then there exits $\zeta \in \Xi$ such that $R\zeta = \xi$. We now show that $R\zeta = N\zeta$. From (4.1),

$$\kappa\left(\mathbf{e}\left(M\sigma_{2n}, N\zeta\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\sigma_{2n}, \zeta\right)\right), \chi\left(\mathrm{Z}\left(\sigma_{2n}, \zeta\right)\right)\right),$$

where

$$\left\|\mathrm{Z}\left(\sigma_{2n}, \zeta\right)\right\|_{\mathbb{E}} = \left\|\frac{\mathbf{e}\left(P\sigma_{2n}, M\sigma_{2n}\right)\mathbf{e}\left(P\sigma_{2n}, N\zeta\right)}{\mathbf{e}\left(P\sigma_{2n}, M\sigma_{2n}\right) + \mathbf{e}\left(P\sigma_{2n}, R\zeta\right) + \mathbf{e}\left(P\sigma_{2n}, N\zeta\right)}\right\|_{\mathbb{E}}.$$

If we take limit as $n \to \infty$, then we get

$$\kappa\left(\left\|\mathbf{e}\left(\xi, N\zeta\right)\right\|_{\mathbb{E}}\right) = \theta \Rightarrow \|\mathbf{e}(\xi, N\zeta)\|_{\mathbb{E}} = 0$$

which implies $\xi = N\zeta$. Therefore we attain that

$$M\upsilon = P\upsilon = \xi = N\zeta = R\zeta,$$

and $C(M, P) \neq \emptyset$, $C(N, R) \neq \emptyset$.

Case ii. Let us suppose that $N\Xi$ is **e**-closed. In this case $\xi \in N\Xi$. As $N\Xi \subseteq P\Xi$, we have $\xi \in P\Xi$ and hence we can choose $\upsilon \in \Xi$ such that $\xi = P\upsilon$. Thus the proof follows as in the same procedure in Case i.

For the cases $M\Xi$ and $R\Xi$ are **e**-closed, the proofs run as in the same Case i and Case ii.

In addition to prove the uniqueness of common fixed point of the mappings, weakly compatibility property have been used. Since the pair $\{M, P\}$ is weakly compatible, we have
$$M\xi = MP\upsilon = PM\upsilon = P\xi.$$
Now on using the inequality (4.1) with $\sigma = \xi$ and $\varsigma = \zeta$, we get

$$\kappa\left(\mathbf{e}\left(M\xi, N\zeta\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\xi, \zeta\right)\right), \chi\left(\mathrm{Z}\left(\xi, \zeta\right)\right)\right),$$

where
$$\mathrm{Z}\left(\xi, \zeta\right) = \frac{\mathbf{e}\left(P\xi, M\xi\right)\mathbf{e}\left(P\xi, N\zeta\right)}{\mathbf{e}\left(P\xi, M\xi\right) + \mathbf{e}\left(P\xi, R\zeta\right) + \mathbf{e}\left(P\xi, N\zeta\right)}.$$

Then it is obvious that $P\xi = M\xi = \xi$, so ξ is a common fixed point of M and P. As the pair $\{N, R\}$ is weakly compatible, we have

$$N\xi = NR\zeta = RN\zeta = R\xi.$$

Now on using the inequality (4.1) with $\sigma = \zeta$ and $\varsigma = \xi$, we get

$$\kappa\left(\mathbf{e}\left(M\xi, N\upsilon\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\xi,\upsilon\right)\right), \chi\left(\mathrm{Z}\left(\xi,\nu\right)\right)\right),$$

where

$$\mathrm{Z}\left(\xi,\upsilon\right) = \frac{\mathbf{e}\left(P\xi, M\xi\right)\mathbf{e}\left(P\xi, N\upsilon\right)}{\mathbf{e}\left(P\xi, M\xi\right) + \mathbf{e}\left(P\xi, R\upsilon\right) + \mathbf{e}\left(P\xi, N\upsilon\right)}.$$

Then $N\xi = R\xi = \xi$, so ξ is a common fixed point of N and R.
Finally we show that M, N, P, R have a unique common fixed point in Ξ. Suppose that ξ and υ are two fixed points. Hence

$$M\xi = N\xi = P\xi = R\xi = \xi,$$

$$M\upsilon = N\upsilon = P\upsilon = R\upsilon = \upsilon.$$

In inequality (4.1) replace σ, ς with ξ and υ, respectively.

$$\kappa\left(\mathbf{e}\left(M\xi, N\upsilon\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\xi,\upsilon\right)\right), \chi\left(\mathrm{Z}\left(\xi,\upsilon\right)\right)\right),$$

where

$$\mathrm{Z}\left(\xi,\upsilon\right) = \frac{\mathbf{e}\left(P\xi, M\xi\right)\mathbf{e}\left(P\xi, N\upsilon\right)}{\mathbf{e}\left(P\xi, M\xi\right) + \mathbf{e}\left(P\xi, R\upsilon\right) + \mathbf{e}\left(P\xi, N\upsilon\right)}.$$

Owing to the properties of function κ, χ and $\mathfrak{F}$ we conclude that $\xi = \upsilon$. Therefore $\xi \in \Xi$ is the unique common fixed point of the mappings M, N, P and R. □

Definition 4.3. Let $(\Xi, \mathbf{e})$ be an elliptic valued metric space, $M, N, P, R : \Xi \to \Xi$ be self mappings. Suppose there exist $\kappa, \chi \in \Upsilon$ and $\mathfrak{F} \in \mathcal{C}$ such that for all $\sigma, \varsigma \in \Xi$ with

$$\kappa\left(\mathbf{e}\left(M\sigma, N\varsigma\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\sigma,\varsigma\right)\right), \chi\left(\mathrm{Z}\left(\sigma,\varsigma\right)\right)\right), \tag{4.16}$$

where

$$\begin{cases} \mathrm{Z}\left(\sigma,\varsigma\right) = \frac{\mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,N\varsigma)+[\mathbf{e}(P\sigma,R\varsigma)]^2+\mathbf{e}(P\sigma,M\sigma)\mathbf{e}(P\sigma,R\varsigma)}{\mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,R\varsigma)+\mathbf{e}(P\sigma,N\varsigma)} \\ \text{for all } \sigma,\varsigma \in \Xi \text{ with } \sigma \neq \varsigma \text{ if } \mathbf{e}\left(P\sigma, M\sigma\right) + \mathbf{e}\left(P\sigma, R\varsigma\right) + \mathbf{e}\left(P\sigma, N\varsigma\right) \neq \theta; \\ \mathbf{e}\left(M\sigma, N\varsigma\right) > \theta, \quad \text{if } \mathbf{e}\left(P\sigma, M\sigma\right) + \mathbf{e}\left(P\sigma, R\varsigma\right) + \mathbf{e}\left(P\sigma, N\varsigma\right) = \theta. \end{cases}$$

Then the pair $\{M, N\}$ is called a generalized $\mathcal{C}_{\mathbb{E}}$-$\{P, R\}$ contractive pair of type II.

Theorem 4.2. *Let* $(\Xi, \mathbf{e})$ *be an* $\mathbf{e}$*-complete elliptic valued metric space and* $M, N, P, R : \Xi \to \Xi$ *be self mappings. Assume that the following statements hold:*

i. the pair $\{M, N\}$ *is generalized* $\mathcal{C}_{\mathbb{E}}$-$\{P, R\}$ *contractive pair of type II;*

ii. $N\Xi \subseteq P\Xi$ *and* $M\Xi \subseteq R\Xi$*;*

iii. one of the ranges $M\Xi, N\Xi, P\Xi$ *and* $R\Xi$ *is* **e***-closed.*

Then $C(M, P) = \{x \in \Xi : x = Mz = Pz\} \neq \emptyset$ *and* $C(N, R) = \{x \in \Xi : x = Nz = Rz\} \neq \emptyset$*. In addition, if* $\{M, P\}$ *and* $\{N, R\}$ *are weakly compatible pairs then the mappings* M, N, P *and* R *have a unique common fixed point in* Ξ*.*

Proof. The proof has the similar line as the proof of Theorem (4.1). □

Definition 4.4. Let $(\Xi, \mathbf{e})$ be an elliptic valued metric space, $M, N, P, R : \Xi \to \Xi$ be self mappings. Suppose there exist $\kappa, \chi \in \Upsilon$ and $\mathfrak{F} \in \mathcal{C}$ such that for all $\sigma, \varsigma \in \Xi$ with

$$\kappa\left(\mathbf{e}\left(M\sigma, N\varsigma\right)\right) \preceq \mathfrak{F}\left(\kappa\left(\mathrm{Z}\left(\sigma,\varsigma\right)\right), \chi\left(\mathrm{Z}\left(\sigma,\varsigma\right)\right)\right), \tag{4.17}$$

where

$$\begin{cases} \mathrm{Z}\left(\sigma,\varsigma\right) = \frac{\mathbf{e}(P\sigma, M\sigma)\mathbf{e}(P\sigma, N\varsigma) + \mathbf{e}(R\varsigma, M\sigma)\mathbf{e}(R\varsigma, N\varsigma)}{\mathbf{e}(P\sigma, M\sigma) + \mathbf{e}(P\sigma, N\varsigma)} \\ \text{for all } \sigma,\varsigma \in \Xi \text{ with } \sigma \neq \varsigma \text{ if } \mathbf{e}\left(P\sigma, M\sigma\right) + \mathbf{e}\left(P\sigma, N\varsigma\right) \neq \theta; \\ \mathbf{e}\left(M\sigma, N\varsigma\right) > \theta, \quad \text{if } \mathbf{e}\left(P\sigma, M\sigma\right) + \mathbf{e}\left(P\sigma, N\varsigma\right) = \theta. \end{cases}$$

Then the pair $\{M, N\}$ is called a generalized $\mathcal{C}_{\mathbb{E}}$-$\{P, R\}$ contractive pair of type III.

Theorem 4.3. *Let* $(\Xi, \mathbf{e})$ *be an* **e***-complete elliptic valued metric space and* $M, N, P, R : \Xi \to \Xi$ *be self mappings. Assume that the following statements hold:*

i. the pair $\{M, N\}$ *is generalized* $\mathcal{C}_{\mathbb{E}}$-$\{P, R\}$ *contractive pair of type III;*

ii. $N\Xi \subseteq P\Xi$ *and* $M\Xi \subseteq R\Xi$*;*

iii. one of the ranges $M\Xi, N\Xi, P\Xi$ *and* $R\Xi$ *is* **e***-closed.*

Then $C(M, P) = \{x \in \Xi : x = Mz = Pz\} \neq \emptyset$ *and* $C(N, R) = \{x \in \Xi : x = Nz = Rz\} \neq \emptyset$*. In addition, if* $\{M, P\}$ *and* $\{N, R\}$ *are weakly compatible pairs then the mappings* M, N, P *and* R *have a unique common fixed point in* Ξ*.*

If we replace $\mathrm{Z}(\sigma,\varsigma)$ with the followings we obtain the same results as above theorems in the setting of elliptic valued metric spaces;

$$\begin{cases} \mathrm{Z}(\sigma,\varsigma) = \frac{[\mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(R\varsigma,N\varsigma)]^2}{\mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,N\varsigma)} \\ \text{for all } \sigma,\varsigma \in \Xi \text{ with } \sigma \neq \varsigma \text{ if } \mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,N\varsigma) \neq \theta; \\ \mathbf{e}(M\sigma,N\varsigma) > \theta, \quad \text{if } \mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,R\varsigma) = \theta. \end{cases}$$

and

$$\begin{cases} \mathrm{Z}(\sigma,\varsigma) = \frac{[\mathbf{e}(P\sigma,M\sigma)]^2+[\mathbf{e}(R\varsigma,N\varsigma)]^2}{\mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,N\varsigma)} \\ \text{for all } \sigma,\varsigma \in \Xi \text{ with } \sigma \neq \varsigma \text{ if } \mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,N\varsigma) \neq \theta; \\ \mathbf{e}(M\sigma,N\varsigma) > \theta, \quad \text{if } \mathbf{e}(P\sigma,M\sigma)+\mathbf{e}(P\sigma,R\varsigma) = \theta. \end{cases}$$

The above results are the generalization of the results existing literature such as ([9]-[13]).

5 Conclusion

Alternative definitions of imaginary unit i, other than $i^2 = -1$, provided the identification of interesting and useful complex number systems. The elliptic valued metric spaces defined in this study, obtained from the elliptic number system with $i^2 = p < 0$, contain complex valued metric spaces defined by Azam et al. in 2011, enabling the study of fixed point theory by making rational and product expressions meaningful. Since $p = -1$ gives the complex number system, the results obtained above for elliptical valued metric spaces are also valid for complex valued metric spaces.

Acknowledgements
The authors would like to thank the anonymous referees for valuable comments that helped to improve this article.

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