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Continuous dependence of solutions for damped improved Boussinesq equation

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Abstract: In this paper, the initial-boundary value problem for a damped nonlinear improved Boussinesq equation is studied. A priori estimates for the solution of the equation are obtained in terms of initial data and coefficients of the problem. The continuous dependence of solutions on dispersive (δ) and (r) and dissipative (b) coefficients are established by multiplier method.

Key words: Nonlinear damped improved Boussinesq equation, continuous dependence, structural stability

1. Introduction

In this study, the following initial boundary value problem for nonlinear damped improved Boussinesq equation is studied:

$$u_{tt} - b\Delta u - \delta\Delta u_{tt} - r\Delta u_t = \Delta f(u), \quad (x,t) \in \Omega \times [0,T], \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \subset \mathbb{R}^n (n \geqslant 3), \tag{1.2}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \quad T > 0, \tag{1.3}$$

where $f(u) = -u|u|^{p-2}$. Here, Δ is a Laplace operator with n dimension, i.e. $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is a differential operator of second order. In this equation, $r > 0$ and $\delta > 0$ are dispersive coefficients, and $b > 0$ is a dissipative coefficient. $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth enough boundary $\partial\Omega$, and $1 < p \leq \infty$ if $n = 1, 2$ and $2 < p \leq \frac{2n-2}{n-2}$ if $n \geq 3$.

Continuous dependence of solutions on parameters of equations is a kind of structural stability that shows the impact of mini changes in parameters of equations on the solutions. The issue of continuous dependence on the parameter has received important attention since 1960 with the result that there are a number of ways for reproducing continuous dependence inequalities for different models described by partial differential equations. Many results of this kind have been extensively investigated by Ames and Straughan [1]. Such works were examined in books [1, 20] and articles [5, 6] and the references therein.

Theoretical and physical modeling of water waves has been studied since the sixteenth century. Generally, it is significant to investigate the waves caused by the movement of an object in the water (ship, boat, etc.), as well as in the open sea and coastal waves caused by wind. Numerical modeling of nonlinear waves is possible with

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Boussinesq equations. In their article, Bayraktar and Beji [2] determined the pressure area that would represent the sailing vessel and the waves that would occur when this pressure area advances were modeled numerically with a computer program. Boussinesq equations are depth-integrated equations and dispersion terms represent the effect of partial fluid acceleration in the vertical direction. Boussinesq equations are distinguished from long wave equations with these properties. Boussinesq equations are generally used to model waves in nearby coastal regions or at medium depth. In addition, Boussinesq equations can be used to model the waves generated by a progressive object. The first Boussinesq model, which is valid for constant water depths, was obtained by Boussinesq [4]. Later, Mei and Meháute [10] and Peregrine [11] also provided Boussinesq equations for unstable water depths. Mei and Meháute omitted the velocity at the base as a variable, while Peregrine used averaged velocity as the variable. Because of the widespread use of the equations derived by Peregrine, these equations are known as standard Boussinesq equations. In order to obtain equations with better dispersion characteristics, Madsen et al. [8] and Madsen and Sørensen [9] added higher-order terms with regular coefficients to Boussinesq equations with constant and variable water depth, respectively. Beji and Nadaoka [3] diversified the improved Boussinesq equations of Madsen et al. [8].

The Boussinesq equation can be written in two fundamental forms as follows:

$$u_{tt} - u_{xx} + \delta u_{xxxx} = (u^2)_{xx}, \tag{1.4}$$

$$u_{tt} - u_{xx} - u_{xxtt} = (u^2)_{xx}. \tag{1.5}$$

Eq. (1.5) is a significant model that approximately details the propagation of long waves in shallow water as in other Boussinesq equations (with u_{xxxx}, instead of u_{xxtt}). Where $\delta > 0$, Eq. (1.4) governs mini nonlinear crosscut oscillations of an elastic beam (see [17] and references therein) and is linearly stable. It is referred to as the ''good'' Boussinesq equation. When $\delta < 0$, it is called the ''bad'' Boussinesq equation because of linear instability. The first person to produce Eq. (1.4) was Boussinesq [4].

Eq. (1.5) is called the "improved" Boussinesq equation (IBq equation). The difference of the improved Boussinesq equation from the Boussinesq equation is that it contains a fourth-order space-time derivative u_{xxtt}. When the application areas of the improved Boussinesq equation are examined, considering the transverse motion and nonlinear conditions, this equation occurs on elastic sticks with circular cross-section within acoustic waves. Also, the bad Boussinesq equation is used to investigate the propagation of ion sound waves in plasma and to investigate nonlinear lattice waves, to approach the bad Boussinesq equation, and to identify waves emitted at right angles to the magnetic field. The good one can be handled in a similar manner.

Scott Russell's research [15] on solitary water waves enabled the advancement of nonlinear partial differential equations to model waves in fluids, plasmas, elastic bodies, etc. Subsequently, Polat et al. [14] found the blow-up of the solutions of the damped Boussinesq equation:

$$u_{tt} - bu_{xx} + \delta u_{xxxx} - ru_{xxt} = f(u)_{xx}. \tag{1.6}$$

Recently, more studies have been done for higher-order Boussinesq equations. Eugene and Schneider [16], who modeled the water wave problem with surface tension, studied a class of Boussinesq equations as follows:

$$u_{tt} - u_{xx} - u_{xxtt} - \mu u_{xxxx} + u_{xxxxtt} = (u^2)_{xx}, \tag{1.7}$$

where $x, t, \mu \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$.

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{1.8}$$

where u_{txx} is the damping term and $\alpha, b > 0$, $\beta \in \mathbb{R}$ are constant values [13]. Varlamov [17] for Eq. (1.8) examined the long-time behavior of solutions to initial boundary value problems in two space dimensions.

Xu et al. [21] examined the blow-up of the solutions of the Cauchy problem for the damped generalized Boussinesq equation and first proved the local existence of a weak solution and a smooth solution, then proved the global existence and finite time blow-up of the solution by using the potential well method and convexity method:

$$u_{tt} - u_{xx} + (u_{xx} + f(u))_{xx} - \alpha u_{xxt} = 0. \tag{1.9}$$

Liu and Wang [7] studied the limit behavior of solutions to the Cauchy problem for the damped Boussinesq equation in the regime of small viscosity in $\mathbb{R}^n$:

$$u^{\epsilon}_{tt} - \Delta u^{\epsilon} + \Delta^2 u^{\epsilon} - \epsilon \Delta u^{\epsilon}_t = \beta \Delta f(u^{\epsilon}), \tag{1.10}$$

with the nonlinear term given by the smooth function $f(u^{\epsilon})$ behaving as for power $f(u^{\epsilon}) = O(|u^{\epsilon}|^p)$ as $u^{\epsilon} \to 0$, where $u^{\epsilon} = u^{\epsilon}(t,x)$ is the unknown function of $x \in \mathbb{R}^n$, $n \geq 3$, and $t > 0$, and the parameters $\epsilon > 0$, $\beta \in \mathbb{R}$ are real constants.

Wang and Su [18] found the global existence and asymptotic behavior of solution of the Cauchy problem for the sixth-order Boussinesq equation with hydrodynamical damped term:

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \Delta u_t = \Delta f(u). \tag{1.11}$$

Wang and Mu [19] proved the existence, uniqueness, and blow-up of the global solution for the Cauchy problem of the multidimensional generalized Boussinesq equation for the special case $\alpha = 2$ in the equation as follows:

$$u_{tt} - \alpha \Delta u_{tt} + \Delta^2 u_{tt} = -\Delta^2 u + \Delta u + \Delta f(u). \tag{1.12}$$

Pişkin and Polat [12] obtained the existence, both locally and globally in time, and the global nonexistence and asymptotic behavior of solutions for the Cauchy problem of a generalized Boussinesq-type equation with a damping term:

$$u_{tt} - \Delta u - \alpha \Delta u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - k \Delta u_t = \Delta f(u), \tag{1.13}$$

in multidimensional form.

- *Cauchy inequality with* ϵ*:*

For any $a, b \geq 0$ and any $\epsilon > 0$ we have the following inequality:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

- *Sobolev embedding theorem:*

Suppose that $1 \leq p \leq n, p^* = \frac{np}{n-p}$, and $u \in W^{1,p}(\mathbb{R}^n)$. Then $u \in L^{p^*}(\mathbb{R}^n)$, and we obtain $C \geq 0$ such that

$$||u||_{L^{p^*}} \leq C||\nabla u||_{L^p}.$$

2. A priori estimates

In this part, a priori estimates of solutions of (1.1) are derived. This will be used to prove the continuous dependence for the parameters.

Theorem 2.1 *For any* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *we obtain a solution* $u \in H_0^1(\Omega)$ *of the problem* (1.1)–(1.3). *Moreover, here the following estimates hold:*

$$||u_t||^2 \leq A_1,\ ||\nabla u||^2 \leq A_2,\ ||\nabla u_t||^2 \leq A_3,\ ||\nabla u_{tt}||^2 \leq A_4, \tag{2.1}$$

where $A_1, A_2, A_3, A_4 > 0$ *are constants based on the initial data and the coefficients of* (1.1).

Proof Multiplying (1.1) by u_t in $L^2(\Omega)$, we get

$$\frac{d}{dt}\Big[\frac{1}{2}||u_t||^2 + \frac{b}{2}||\nabla u||^2 + \frac{\delta}{2}||\nabla u_t||^2 + \frac{1}{p}||u||_p^p\Big] + r||\nabla u_t||^2 = 0. \tag{2.2}$$

It follows from (2.2) that

$$E_u(t) = \frac{1}{2}||u_t||^2 + \frac{\sigma}{2}||\nabla u||^2 + \frac{m^2}{2}||u||^2 + \frac{\lambda}{p+1}||u||_{p+1}^{p+1} \leq E_u(0). \tag{2.3}$$

Hence, (2.1) follows from (2.2). From (2.2) it is also known that

$$\frac{d}{dt}E_u(t) + r||\nabla u_t||^2 \leq 0,$$

and we integrate this over $[0, t]$ and then we find (2.1) since $E_u(t) > 0$. □

3. Continuous dependence on coefficients

In this part, it will be shown that the solution of the problem (1.1)–(1.3) depends continually on the coefficients, which are b and δ.

3.1. Continuous dependence on the damping term δ

Suppose that u and v are the solution of (1.1)–(1.3):

$$u_{tt} - b\Delta u - \delta_1 \Delta u_{tt} - r\Delta u_t + u|u|^{p-2} = 0, \quad (x,t) \in \Omega \times [0,T],$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \subset \mathbb{R}^n (n \geqslant 3),$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \quad T > 0.$$

$$v_{tt} - b\Delta v - \delta_2 \Delta v_{tt} - r\Delta v_t + v|v|^{p-2} = 0, \quad (x,t) \in \Omega \times [0,T],$$

$$v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in \Omega \subset \mathbb{R}^n (n \geqslant 3)$$

$$v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \quad T > 0.$$

The difference $w = u - v$ and $\delta = \delta_1 - \delta_2$ of the solutions of these problems is the solution of the initial boundary value problem as follows:

$$w_{tt} - b\Delta w - \delta_1 \Delta w_{tt} - \delta \Delta v_{tt} - r\Delta w_t + (|u|^{p-2}u - |v|^{p-2}v) = 0 \quad (x,t) \in \Omega \times [0,T], \tag{3.1}$$

$$w(x,0) = 0, \qquad w_t(x,0) = 0 \qquad in\ \Omega, \tag{3.2}$$

$$w(x,t) = 0 \quad on \quad \partial\Omega \times (0,\infty). \tag{3.3}$$

Theorem 3.1 *The solution w of problem (3.1)–(3.3) supplies the following inequality:*

$$\frac{1}{2}||w_t||^2 + \frac{b}{2}||\nabla w||^2 + \frac{\delta_1}{2}||\nabla w_t||^2 \leq \frac{e^{M_1 t} A_4}{r M_1}\delta^2 \qquad \forall t > 0, \tag{3.4}$$

where $A_4 > 0$, $M_1 > 0$ are constants based on the parameters and initial data of (1.1).

Proof Multiplying (3.1) by w_t in $L^2(\Omega)$, we obtain

$$\frac{d}{dt}\Big[\frac{1}{2}||w_t||^2 + \frac{b}{2}||\nabla w||^2 + \frac{\delta_1}{2}||\nabla w_t||^2\Big] + r||\nabla w_t||^2 + \delta(\nabla v_{tt}, \nabla w_t) + \\ + \int_\Omega (|u|^{p-2}u - |v|^{p-2}v) w_t dx = 0, \tag{3.5}$$

$$\frac{d}{dt}E_w(t) + r||\nabla w_t||^2 \leq -\delta(\nabla v_{tt}, \nabla w_t) - \int_\Omega (|u|^{p-2}u - |v|^{p-2}v) w_t dx, \tag{3.6}$$

where

$$E_w(t) = \frac{1}{2}||w_t||^2 + \frac{b}{2}||\nabla w||^2 + \frac{\delta_1}{2}||\nabla w_t||^2.$$

Using the Cauchy inequality and Cauchy–Schwarz inequality with ϵ, where $\epsilon = \frac{r}{4}$, the following is obtained:

$$\delta||\nabla v_{tt}||||\nabla w_t|| \leq \frac{r}{4}||\nabla w_t||^2 + \frac{\delta^2}{r}||\nabla v_{tt}||. \tag{3.7}$$

Notice that, after using mean value theorem and Hölder and Sobolev inequalities, respectively, the following is derived:

$$\begin{aligned} \Big|\int_\Omega \Big(|u|^{p-2}u - |v|^{p-2}v\Big) w_t dx\Big| &\leq (p-1)\int_\Omega |w||w_t|\Big(|u|^{p-2} + |v|^{p-2}\Big)dx \\ &\leq (p-1)||w_t||||w||_{\frac{2n-2}{n-2}}\Big(||u||^{p-2}_{(p-2)n} + ||v||^{p-2}_{(p-2)n}\Big) \\ &\leq (p-1)||w_t||C_1||\nabla w||C_2\Big(||\nabla u||^{p-2} + ||\nabla v||^{p-2}\Big). \end{aligned} \tag{3.8}$$

Putting all of these estimates into inequality (3.6) and considering (2.1), we obtain

$$\frac{d}{dt}E_w(t) + r||\nabla w_t||^2 \leq \epsilon||\nabla w_t||^2 + \frac{\delta^2}{4\epsilon}||\nabla v_{tt}||^2 + (p-1)]C_1||w_t||||\nabla w||C_2A_2^{\frac{p-2}{2}}$$
$$\leq \frac{r}{4}||\nabla w_t||^2 + \frac{\delta^2}{r}||\nabla v_{tt}||^2 + C_3||w_t||||\nabla w||, \tag{3.9}$$

where $\epsilon = \frac{r}{4}, \quad C_3 = 2(p-1)C_1C_2A_2^{\frac{p-2}{2}}$. Inequality (3.9) implies

$$\frac{d}{dt}E_w(t) \leq M_1E_w(t) + \frac{\delta^2}{r}||\nabla v_{tt}||^2, \tag{3.10}$$

where $r > 0, ||\nabla w_t||^2 > 0, M_1 = max\left\{1, C_3, b, \frac{\delta_1}{2}, \frac{r}{4}\right\}$. Multiplying (3.10) by e^{-M_1t} in $L^2(\Omega)$ from (2.1), the desired result is found:

$$E_w(t) \leq \frac{\delta^2 A_4}{rM_1}e^{M_1t}. \tag{3.11}$$

□

3.2. Continuous dependence on the coefficient b

Suppose that u and v are the solution of (1.1)–(1.3):

$$u_{tt} - b_1\Delta u - \delta\Delta u_{tt} - r\Delta u_t + u|u|^{p-2} = 0 \quad (x,t) \in \Omega \times [0,T],$$
$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \subset \mathbb{R}^n (n \geqslant 3),$$
$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \quad T > 0.$$

$$v_{tt} - b_2 v - \delta\Delta v_{tt} - r\Delta v_t + v|v|^{p-2} = 0, \quad (x,t) \in \Omega \times [0,T]$$
$$v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in \Omega \subset \mathbb{R}^n (n \geqslant 3),$$
$$v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \quad T > 0.$$

The difference $w = u - v$ and $b = b_1 - b_2$ of the solutions of these problems is the solution of the initial boundary value problem as follows:

$$w_{tt} - b_1\Delta w - b\Delta v - \delta\Delta w_{tt} - r\Delta w_t + (|u|^{p-2}u - |v|^{p-2}v) = 0 \quad (x,t) \in \Omega \times [0,T], \tag{3.12}$$

$$w(x,0) = 0, \qquad w_t(x,0) = 0 \qquad in\ \Omega, \tag{3.13}$$

$$w(x,t) = 0 \quad on \quad \partial\Omega \times (0,\infty). \tag{3.14}$$

Theorem 3.2 *The solution w of problem (3.12)–(3.14) satisfies the inequality*

$$\frac{1}{2}||w_t||^2 + \frac{b_1}{2}||\nabla w||^2 + \frac{\delta}{2}||\nabla w_t||^2 \leq \frac{A_2 b^2}{r M_1} e^{M_1 t} \qquad \forall t > 0, \tag{3.15}$$

where $A_2 > 0$, $M_1 > 0$ are constants that depend on the parameters and initial data of (1.1).

Proof Multiplying (3.12) by w_t in $L^2(\Omega)$, we obtain

$$\begin{aligned} \frac{d}{dt}E_w(t) + r||\nabla w_t||^2 &\leq -b(\nabla v, \nabla w_t) - \int_\Omega (|u|^{p-2}u - |v|^{p-2}v) w_t dx \\ &\leq \frac{r}{4}||\nabla w_t||^2 + \frac{b^2}{r}||\nabla v||^2 + C_3\frac{||w_t||^2}{2} + C_3\frac{||\nabla w||^2}{2}, \end{aligned}$$

where

$$E_w(t) = \frac{1}{2}||w_t||^2 + \frac{b_1}{2}||\nabla w||^2 + \frac{\delta}{2}||\nabla w_t||^2.$$

Then,

$$\frac{d}{dt}E_w(t) \leq \frac{b^2}{r}||\nabla v||^2 + M_2 E_w(t),$$

where $M_2 = max\left\{1, \frac{\delta}{2}, b_1, C_3, \frac{r}{4}\right\}$.
That is,

$$E_w(t) \leq \frac{A_2 b^2}{r M_1} e^{M_2 t},$$

which indicates continuous dependency on b. The proof is completed. □

Remark 3.1 Besides the above approach, continuous dependency on the coefficient r and for all coefficients can also be studied in similarly proved calculations for the other coefficients.

Conclusion

In this article, using the multiplier method, we conclude that the solution of the problem (1.1)–(1.3) describing a damped improved Boussinesq equation is continuously dependent on the damping term δ and dissipative coefficient b. Hence, the effects of small perturbations of parameters on solutions are obtained.

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