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$\mathcal{I}_2$-LACUNARY STRONGLY SUMMABILITY FOR MULTIDIMENSIONAL MEASURABLE FUNCTIONS

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Abstract. Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a nontrivial ideal. We provide a new approach to the concept of $\mathcal{I}_2$-double lacunary statistical convergence and $\mathcal{I}_2$-lacunary strongly double summable by taking $f(\tau, \upsilon)$, which is a multidimensional measurable real valued function on $(1, \infty) \times (1, \infty)$. Additionally, we examine the relation between these two new methods.

1. Introduction

The concept of a statistical convergence was introduced by Fast [**9**], and Steinhaus [**30**] independently in the same year 1951. Actually, the idea of statistical convergence was used to proved theorems on the statistical convergence of Fourier series by Zygmund [**31**] in the first edition of his celebrated monograph published in Warsaw. He used the term "*almost convergence*" place of statistical convergence and at that time this idea was not recognized much. Since the term "*almost convergence*" was already in use Lorentz [**18**], Fast [**9**] had to choose a different name for his concept and "*statistical convergence*" was mostly the suitable one. Active research on this topic started after the paper of Fridy [**10**] and since then a large collection of literature has appeared. At the last quarter of the 20th century, statistical convergence has been discussed and captured important aspect in creating the basis of several investigations conducted in main branches of mathematics such as the theory of number [**7**], measure theory [**19**], trigonometric series [**31**], probability theory [**6**], and approximation theory [**12**]. In addition, it was further investigated from the sequence space point of view and linked with summability theory by Connor [**4**], Et at. al. [**8**], Kolk [**13**], Orhan et al. [**11**], Kumar and Mursaleen [**15**], Rath and Tripathy [**24**], Šalát [**25**], and many others made substantial contributions to the theory.

Definition 1.1. Let R be a subset of $\mathbb{N}$ and $R_m = \{i \leqslant m : i \in R\}$. The natural density of R is defined $\delta(R) = \lim_m \frac{1}{m}|R_m|$ provided it exists. Here, and in

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what follows, $|R_m|$ denotes the cardinality of set R_m. A sequence $y = (y_i)$ is said to be statistically convergent to the number L, provided that for every $\varepsilon > 0$, the set $R_\varepsilon = \{i \in \mathbb{N} : |y_i - L| \geqslant \varepsilon\}$ has natural density zero, that is

$$\lim_{m\to\infty} |\{i \leqslant m : |y_i - L| \geqslant \varepsilon\}| = 0.$$

Whenever this occurs, we can write $st - \lim_i y_i = L$.

In 1993, Fridy and Orhan [**11**] established the following relation between lacunary statistical convergence and statistical convergence.

Definition 1.2. By a lacunary sequence $\theta = (p_r)$, $r = 0, 1, 2, \ldots$ where k_0, we shall mean an increasing sequence of nonnegative integers with $p_r - p_{r-1} \to \infty$. The intervals determined by θ will be denoted by $J_s = (p_{r-1}, p_r]$ and $z_r = p_r - p_{r-1}$. The ratio $\frac{p_r}{p_{r-1}}$ will be denoted by q_r. Let $\theta = (p_r)$ be a lacunary sequence; the number sequence y is S_θ-convergent to L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{p_r} |\{i \in J_s : |y_i - L| \geqslant \varepsilon\}| = 0.$$

In this case, we write $S_\theta - \lim_{i\to\infty} y_i = L$ or $y_i \to L(S_\theta)$.

In 1970, Bernstein [**3**] introduced convergence of sequences with respect to a filter $\mathcal{F}$ on $\mathbb{N}$. Using the concept of an ideal, the idea statistical convergence was further extended to $\mathcal{I}$-convergence in [**14**]. The ideal convergence provides a general framework to study the properties of various types of convergence. Some of the most important applications of ideals can be found in [**16, 17, 26, 27**].

For any nonempty set Y, $\mathcal{P}(Y)$ denotes the power set of Y. A family of sets $\mathcal{I} \subset \mathcal{P}(Y)$ is said to be an "*ideal*" on Y if and only if

(i) $\emptyset \in \mathcal{I}$;
(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$;
(iii) For each $A \in \mathcal{I}$ and $B \subseteq A$ we have $B \in \mathcal{I}$.

A nonempty family of sets $\mathcal{F} \subset \mathcal{P}(Y)$ is said to be "*filter*" on Y if and only if

(i) $\emptyset \notin \mathcal{F}$;
(ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
(iii) For $A \in \mathcal{F}$ and $B \supseteq A$ we have $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ on Y is called "*nontrivial*" if $\mathcal{I} \neq \emptyset$ and $Y \notin \mathcal{I}$. It is clear that $\mathcal{I} \subset \mathcal{P}(Y)$ is a nontrivial ideal on Y if and only if $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{Y - A : A \in \mathcal{I}\}$ is a filter on Y. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$. A nontrivial ideal $\mathcal{I} \subset \mathcal{P}(Y)$ is called an admissible ideal in Y if and only if it contains all singletons i.e. if it contains $\{\{y\} : y \in Y\}$.

Using the above terminology, Kostyrko et al. [**14**] defined $\mathcal{I}$-convergence in a metric space as follows:

Definition 1.3. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a nontrivial ideal in $\mathbb{N}$ and (Y, d) be a metric space. A sequence $y = (y_i)$ in Y is said to be $\mathcal{I}$-convergent to ψ if for each $\varepsilon > 0$, then the set

$$A(\varepsilon) = \{i \in \mathbb{N} : d(y_i, \psi) \geqslant \varepsilon\} \in \mathcal{I}.$$

Under this condition, we write $\mathcal{I} - \lim_{i\to\infty} y_i = \psi$.

Recently, Kostyrko et al. in [**14**] and Savas and Gumus [**26**] introduced new concept of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence respectively. In recent years, ideas of statistical convergence, lacunary statistical convergence and $\mathcal{I}$-convergence have been respectively extended from single to double sequence in [**2, 17**], and [**28**].

We now present the following definitions, which will be needed in the sequel.

DEFINITION 1.4. [**23**] A double sequence $y = (y_{i,j})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim *sense* if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|y_{i,j} - L| < \varepsilon$, whenever $i, j > N_\varepsilon$. In this case, we denote such limit as follow: $P - \lim_{i,j\to\infty} y_{i,j} = L$ and $y \xrightarrow{P} L$.

The following concept of statistical convergence for double sequences was presented by Mursaleen and Edely [**20**]. Also, Savaş and Patterson [**28**] introduced the notion of double lacunary sequence and defined the lacunary statistical convergence for double sequence, and please note that let

$$\mathcal{I}_0 = \{A \subset \mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geqslant m(A) \Rightarrow (i, j) \notin A)\}.$$

Then $\mathcal{I}_0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is admissible if and only if $\mathcal{I}_0 \subset \mathcal{I}_2$. Additionally, if $\mathcal{I}_2$ is the $\mathcal{I}_0$, then $\mathcal{I}_2$-convergence coincides with the convergence in Pringsheim's sense and if we take

$$\mathcal{I}_d = \{A \subset \mathbb{N}\times\mathbb{N} : \delta_2(A) = 0\},$$

then $\mathcal{I}_d$-convergence becomes statistical convergence for double sequences [**2**]. While the work on sequences continued, strongly summable functions were introduced by Borwein [**1**]. Following Borwein's results, in 2010, Nuray [**21**] introduced λ-strongly summable and λ-statistically convergent functions by taking real valued Lebesgue measurable function on $(1, \infty)$. Recently, Connor and Savaş [**5**] introduced lacunary statistical and sliding window convergence for measurable functions. In [**22**], Nuray and Aydin introduced lacunary strongly convergence, statistical convergence and lacunary statistical convergence of measurable functions on interval $(1, \infty)$. In 2019, by using Pringsheim limits, Savas [**29**] presented the new notion of multidimensional strongly Cesáro type Summability method by taking a real valued measurable functions $f(\tau, \upsilon)$ defined on $(1, \infty) \times (1, \infty)$ as follows:

A function $f(\tau, \upsilon)$ is said to be strongly double Cesáro summable to L if

$$P - \lim_{m,n\to\infty} \frac{1}{mn}\int_1^m \int_1^n |f(\tau, \upsilon) - L| d\tau\, d\upsilon = 0.$$

The space of all strongly double Cesáro summable functions will be denoted by $[W]_2$.

Following Savas's results, in this paper we will present the more general notion of $\mathcal{I}_2$-lacunary double statistical convergence and $\mathcal{I}_2$-lacunary strongly double summability by taking nonnegative multidimensional measurable real valued function on $(1, \infty) \times (1, \infty)$. Moreover, we will establish the relationship between two summability methods.

2. Main Results

In this section, we shall present the following new definitions. Additionally, the relationship between these concepts are investigated. Throughout this paper $f(\tau, \upsilon)$ shall be a multidimensional measurable real valued function on $(1, \infty) \times (1, \infty)$.

Definition 2.1. A function $f(\tau, \upsilon)$ is said to be *statistically double bounded* if there exists some constant M such that

$$P - \lim_{m,n \to \infty} \frac{1}{mn} |\{(\tau, \upsilon) : \tau \leqslant m, \upsilon \leqslant n : |f(\tau, \upsilon| \geqslant M\}| = 0,$$

where the vertical bars indicate the Lebesgue measure of the enclosed set. We will denote the set of all double bounded double functions by $F(\ell)^2_\infty$.

Now, we will define the definition of double lacunary function to present our main definitions of this paper.

Definition 2.2. The double function $\Theta_F = \{g(t), h(s)\}$ is called double lacunary function if there exist two increasing functions such that

$$g(0) = 0, \ \alpha(t) = g(t) - g(t-1) \to \infty \text{ as } t \to \infty,$$
$$h(0) = 0, \ \beta(s) = h(s) - h(s-1) \to \infty \text{ as } s \to \infty.$$

where $\frac{g(t)}{g(t+1)} \leqslant 1$, $\frac{h(s)}{h(s+1)} \leqslant 1$, and $\frac{g(r)}{h(r)} \leqslant 1$ because of $g(1) \leqslant h(1) \leqslant g(2) \leqslant h(2) \leqslant \ldots \leqslant g(r-1) \leqslant h(r-1) \leqslant g(r) \leqslant h(r)$ as $r \to \infty$. We shall use the following notations in the sequel, $g(t, s) = g(t) \cdot h(s)$ and $\alpha(t, s) = \alpha(t) \cdot \beta(s)$, Θ_F is determined by $I_{t,s} = \{(\tau, \upsilon) : g(t-1) < \tau \leqslant g(t) \ \& \ h(s-1) < \upsilon \leqslant h(s)\}$, $\xi(t) = \frac{g(t)}{g(t-1)}$ and $\varphi(s) = \frac{h(s)}{h(s-1)}$, $\xi(t, s) = \xi(t) \cdot \varphi(s)$.

Definition 2.3. Let us consider the double lacunary function $\Theta_F = \{g(t), h(s)\}$. A function $f(\tau, \upsilon)$ is said to be lacunary double statistically convergent to L if for each $\varepsilon > 0$,

$$P - \lim_{t,s \to \infty} \frac{1}{\alpha(t, s)} |\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - L| \geqslant \varepsilon\}|,$$

where the vertical bars indicate the Lebesgue measure of the enclosed set. Whenever this occurs, we write $S_{\Theta_F} - \lim f(\tau, \upsilon) = L$. The set of all lacunary double statistically convergent functions will be denoted by $[S_{\Theta_F}]$.

Definition 2.4. Let us consider the ordered pair of double lacunary functions $\Theta_F = \{g(t), h(s)\}$. A function $f(\tau, \upsilon)$ is said to be lacunary strongly double summable to L, if

$$P - \lim_{t,s \to \infty} \frac{1}{\alpha(t, s)} \int_{g(t-1)}^{g(t)} \int_{h(s-1)}^{h(s)} |f(\tau, \upsilon) - L| d\tau \, d\upsilon = 0.$$

Whenever this occurs, we write $[N_{\Theta_F}] - \lim f(\tau, \upsilon) = L$ and

$$[N_{\Theta_F}] = \left\{ f(\tau, \upsilon) : \exists \text{ some } L, \ P - \lim_{t,s \to \infty} \frac{1}{\alpha(t, s)} \iint\limits_{(\tau,\upsilon) \in I_{t,s}} |f(\tau, \upsilon) - L| d\tau \, d\upsilon = 0 \right\}.$$

We shall denote the set of all lacunary strongly double summable functions by $[N_{\Theta_F}]$.

EXAMPLE 2.1. Let us consider the double lacunary function ordered pair of functions $\Theta_F = \{g(t), h(s)\}$ and $f(\tau, \upsilon)$ define as follows:

$$f(\tau,\upsilon) = \begin{cases} \frac{1}{\alpha(t,s)} \operatorname{sgn}(\gamma(\tau,\upsilon)), & \text{if } (\tau,\upsilon) \in I_{t,s} \\ 0, & \text{if otherwise} \end{cases}$$

where $\gamma(\tau, \upsilon)$ denote the collection of functions that are bounded on $I_{t,s}$, note that $f(\tau, \upsilon) \in [N_{\Theta_F}]$.

DEFINITION 2.5. Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a nontrivial ideal. A function $f(\tau, \upsilon)$ is said to be $\mathcal{I}_2$-convergent in Pringsheim sense to a number L, if for every $\varepsilon > 0$,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\} \in \mathcal{I}_2.$$

Whenever this occurs, we write $\mathcal{I}_2 - \lim_{\tau,\upsilon\to\infty} f(\tau,\upsilon) = L$.

DEFINITION 2.6. Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a nontrivial ideal. A function $f(\tau, \upsilon)$ is said to be $\mathcal{I}_2$-double statistical convergent or $S_F^2(\mathcal{I}_2)$-convergent to L, if for each $\varepsilon > 0$ and $\delta > 0$,

$$\Big\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{\tau \leqslant m, \upsilon \leqslant n : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}| \geqslant \delta\Big\} \in \mathcal{I}_2.$$

In this case, we write $S_F^2(\mathcal{I}_2) - \lim_{\tau,\upsilon\to\infty} f(\tau,\upsilon) = L$ or $f(\tau,\upsilon) \xrightarrow{P} L(S_F^2(\mathcal{I}_2))$, where $S_F^2(\mathcal{I}_2)$ denotes the set of all $\mathcal{I}_2$-double statistically convergent functions.

DEFINITION 2.7. Let us consider the double lacunary function ordered pair of functions $\Theta_F = \{g(t), h(s)\}$ and $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a nontrivial ideal. A function $f(\tau, \upsilon)$ is said to be $\mathcal{I}_2$-double lacunary statistically convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$,

$$\Big\{(t,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\alpha(t,s)} |\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}| \geqslant \delta\Big\} \in \mathcal{I}_2.$$

In this case, we write $f(\tau,\upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$ or $S_{\Theta_F}(\mathcal{I}_2) - \lim_{\tau,\upsilon\to\infty} f(\tau,\upsilon) = L$, where $S_{\Theta_F}(\mathcal{I}_2)$ denotes the set of all $\mathcal{I}_2$-lacunary double statistically convergent functions.

DEFINITION 2.8. Let us consider the double lacunary function $\Theta_F = \{g(t), h(s)\}$ and $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a nontrivial ideal. A function $f(\tau, \upsilon)$ is said to be $N_{\Theta_F}(\mathcal{I}_2)$-lacunary strongly double summable to L, if for every $\varepsilon > 0$ we have,

$$\Big\{(t,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\alpha(t,s)} \iint_{(\tau,\upsilon)\in I_{t,s}} |f(\tau,\upsilon) - L| \geqslant \varepsilon\Big\} \in \mathcal{I}_2.$$

When this occurs, we write $f(\tau,\upsilon) \xrightarrow{P} L(N_{\Theta_F}(\mathcal{I}_2))$ or $N_{\Theta_F}(\mathcal{I}_2) - \lim_{\tau,\upsilon\to\infty} f(\tau,\upsilon) = L$. $N_{\Theta_F}(\mathcal{I}_2)$ denotes the set of all $N_{\Theta_F}(\mathcal{I}_2)$-lacunary strongly double summable functions.

EXAMPLE 2.2. If we take

$\mathcal{I}_2 = \{K \subset \mathbb{N}\times\mathbb{N} : K = (\mathbb{N}\times R)\cup(R\times\mathbb{N}) \text{ for some finite subset } R \text{ of } \mathbb{N}\}.$

Let $g(t) = (2^t)$ and $h(s) = (3^s)$ be two lacunary functions. We take a special set $A \in \mathcal{I}_2$ and define a real valued function $f(\tau, \upsilon)$ by

$$f(\tau,\upsilon) = \begin{cases} \sqrt{\tau\upsilon}, & \text{for } (t,s) \notin A,\ 2^{t-1}+1 \leqslant t \leqslant 2^t + \sqrt{\alpha(t)} \text{ and} \\ & \qquad\qquad\quad 3^{s-1}+1 \leqslant s \leqslant 3^s + \sqrt{\beta(s)}, \\ \tau\upsilon, & \text{for } (t,s) \in A,\ 2^{t-1} < t \leqslant 2^t + (\alpha(t))^2 \text{ and} \\ & \qquad\qquad\quad 3^{s-1} < r \leqslant 3^s + (\beta(s))^2 \\ 0, & \text{otherwise.} \end{cases}$$

where $I_t = (2^{t-1}, 2^t]$ and $I_s = (3^{s-1}, 3^s]$. Then for each $\varepsilon > 0$, we have

$$P-\lim_{t,s\to\infty} \frac{1}{\alpha(t,s)} |\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - 0| \geqslant \varepsilon\}| \leqslant P-\lim_{t,s\to\infty} \frac{\sqrt{\alpha(t)}\sqrt{\beta(s)}}{(\alpha(t,s))} = 0,$$

for $(t,s) \neq A$. For $\delta > 0$, there exists a positive integer z_0 such that

$$\frac{1}{\alpha(t,s)} |\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - 0| \geqslant \varepsilon\}| < \delta,$$

for every $(t,s) \notin A$ and $t,s \geqslant z_0$. Let $B = \{1,2,\ldots,z_0-1\}$ and

$$E = \Big\{(t,s) \notin A : \frac{1}{\alpha(t,s)} |\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - 0| \geqslant \varepsilon\}| \geqslant \delta\Big\}.$$

Thus, $E \subseteq (\mathbb{N}\times B)\cup(B\times\mathbb{N})$ and $E \in \mathcal{I}_2$ by structure of the ideal $\mathcal{I}_2$. Therefore

$$\Big\{(t,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)} |\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - 0| \geqslant \varepsilon\}| \geqslant \delta\Big\} \subset A \cup E.$$

It follows that $S_{\Theta_F}(\mathcal{I}_2) - \lim_{\tau,\upsilon\to\infty} f(\tau,\upsilon) = 0$. However, similarly

$$P-\lim_{t,s\to\infty} \frac{1}{\alpha(t,s)} |(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - 0| \geqslant \varepsilon| \nrightarrow 0.$$

This example demonstrates that $S_{\Theta_F}(\mathcal{I}_2)$-double statistical convergence is a generation of S_{Θ_F}-double statistical convergence for the functions.

THEOREM 2.1. *Let $\mathcal{I}_2 \subset \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be an admissible ideal and $\Theta_F = \{g(t), h(s)\}$ be a double lacunary function. Then we have the following:*

(1) *$f(\tau,\upsilon) \xrightarrow{P} L(N_{\Theta_F}(\mathcal{I}_2))$ implies $f(\tau,\upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$;*
(2) *$N_{\Theta_F}(\mathcal{I}_2)$ is a proper subset of $S_{\Theta_F}(\mathcal{I}_2)$;*
(3) *If $f(\tau,\upsilon)$ is statistically bounded and $f(\tau,\upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$ then $f(\tau,\upsilon) \xrightarrow{P} L(N_{\Theta_F}(\mathcal{I}_2))$.*

PROOF. (1) Suppose $f(\tau,\upsilon) \xrightarrow{P} L(N_{\Theta_F}(\mathcal{I}_2))$. For $\varepsilon > 0$, we can write

$$\iint_{(\tau,\upsilon)\in I_{t,s}} |f(\tau,\upsilon) - L|\, d\tau\, d\upsilon \geqslant \iint_{(\tau,\upsilon)\in I_{t,s}, |f(\tau,\upsilon)-L|\geqslant\varepsilon} |f(\tau,\upsilon) - L|\, d\tau\, d\upsilon$$
$$\geqslant \varepsilon |\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}|;$$

which implies

$$\frac{1}{\varepsilon\alpha(t,s)} \iint\limits_{(\tau,\upsilon)\in I_{t,s}} |f(\tau,\upsilon)-L|d\tau\, d\upsilon \geqslant \frac{1}{\alpha(t,s)}|\{(\tau,\upsilon)\in I_{t,s} : |f(\tau,\upsilon)-L| \geqslant \varepsilon\}|.$$

Hence, for any $\delta > 0$, we have the containment

$$\Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}|\{(\tau,\upsilon)\in I_{t,s} : |f(\tau,\upsilon)-L| \geqslant \varepsilon\}| \geqslant \delta\Big\}$$
$$\subseteq \Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)} \iint\limits_{(\tau,\upsilon)\in I_{t,s}} |f(\tau,\upsilon)-L|d\tau\, d\upsilon \geqslant \varepsilon\delta\Big\}.$$

Since $f(\tau,\upsilon) \xrightarrow{P} L(N_{\Theta_F}(\mathcal{I}_2))$, it follows that the later set belongs to $\mathcal{I}_2$ and thus

$$\Big\{(t,s\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}|\{(\tau,\upsilon)\in I_{t,s} : |f(\tau,\upsilon)-L| \geqslant \varepsilon\}| \geqslant \delta\Big\} \in \mathcal{I}_2.$$

Therefore, $f(\tau,\upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$.

(2) Let $f = f(\tau,\upsilon)$ be defined as follows:

$$f(\tau,\upsilon) = \begin{pmatrix} 1 & 2 & 3 & \cdots & \sqrt[3]{\alpha(t,s)} & 0 & \cdots \\ 2 & 2 & 3 & \cdots & \sqrt[3]{\alpha(t,s)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sqrt[3]{\alpha(t,s)} & \sqrt[3]{\alpha(t,s)} & \cdots & \cdots & \sqrt[3]{\alpha(t,s)} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is clear that $f(\tau,\upsilon)$ is an unbounded double function and for $\varepsilon > 0$,

$$\frac{1}{\alpha(t,s)}|\{(\tau,\upsilon)\in I_{t,s} : |f(\tau,\upsilon)-0| \geqslant \varepsilon\}| \leqslant \frac{\sqrt[3]{\alpha(t,s)}}{\alpha(t,s)}$$

which implies for any $\delta > 0$, the containment

$$\Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}|\{(\tau,\upsilon)\in I_{t,s} : |f(\tau,\upsilon)| \geqslant \varepsilon\}| \geqslant \delta\Big\}$$
$$\subseteq \Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{\sqrt[3]{\alpha(t,s)}}{\alpha(t,s)} \geqslant \delta\Big\}.$$

Since $P - \lim \frac{\sqrt[3]{\alpha(t,s)}}{\alpha(t,s)} = 0$. It follows that the set on the right side is finite and therefore belongs to $\mathcal{I}_2$. This shows that

$$\Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}|\{(\tau,\upsilon)\in I_{t,s} : |f(\tau,\upsilon)| \geqslant \varepsilon\}| \geqslant \delta\Big\} \in \mathcal{I}_2,$$

and thus we obtain $f(\tau,\upsilon) \xrightarrow{P} 0\ (S_{\Theta_F}(\mathcal{I}_2))$. On the other hand

$$\frac{1}{\alpha(t,s)} \iint_{(\tau,\upsilon)\in I_{t,s}} |f(\tau,\upsilon)| d\tau\, d\upsilon = \frac{\sqrt[3]{\alpha(t,s)}(\sqrt[3]{\alpha(t,s)}(\sqrt[3]{\alpha(t,s)}+1))}{2\alpha(t,s)} \xrightarrow{P} \frac{1}{2} \text{ as } t,s\to\infty$$

implies that the function

$$\Big(\frac{1}{\alpha(t,s)}\sqrt[3]{\alpha(t,s)}\big(\sqrt[3]{\alpha(t,s)}\big(\sqrt[3]{\alpha(t,s)}+1\big)\big)\Big) \xrightarrow{P} 1 \text{ as } t,s\to\infty$$

and for $\varepsilon = \frac{1}{4}$, we are granted the following

$$\begin{aligned} &\Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)} \iint (\tau,\upsilon)\in I_{t,s} |f(\tau,\upsilon)| d\tau\, d\upsilon \geqslant \frac{1}{4}\Big\} \\ &\quad = \Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}\sqrt[3]{\alpha(t,s)}(\sqrt[3]{\alpha(t,s)}(\sqrt[3]{\alpha(t,s)}+1)) \geqslant \frac{1}{2}\Big\} \in \mathcal{F}(\mathcal{I}_2). \end{aligned}$$

This shows that $f(\tau,\upsilon) \xrightarrow{P} 0\ (N_{\Theta_F}(\mathcal{I}_2))$ does not hold.

(3) Provided that $f(\tau,\upsilon) \in F(\ell)^2_\infty$ such that $f(\tau,\upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$. Then there exists a $R > 0$ such that $|f(\tau,\upsilon) - L| \leqslant R$ for all $(\tau,\upsilon)\in\mathbb{N}\times\mathbb{N}$. Also for each $\varepsilon > 0$, we can write

$$\begin{aligned} \frac{1}{\alpha(t,s)} \iint_{(\tau,\upsilon)\in I_{t,s}} |f(\tau,\upsilon) - L| d\tau\, d\upsilon &= \frac{1}{\alpha(t,s)} \iint_{(\tau,\upsilon)\in I_{t,s}, |f(\tau,\upsilon)-L|\geqslant\frac{\varepsilon}{2}} |f(\tau,\upsilon) - L| d\tau\, d\upsilon \\ &\quad + \frac{1}{\alpha(t,s)} \iint_{(\tau,\upsilon)\in I_{t,s}, |f(\tau,\upsilon)-L|\leqslant\frac{\varepsilon}{2}} |f(\tau,\upsilon) - L| d\tau\, d\upsilon \\ &\leqslant \frac{R}{\alpha(t,s)}\Big|\Big\{(\tau,\upsilon)\in I_{t,s} : |f(\tau,\upsilon) - L| \geqslant \frac{\varepsilon}{2}\Big\}\Big| + \frac{\varepsilon}{2}. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} &\Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)} \iint_{(\tau,\upsilon)\in I_{t,s}} |f(\tau,\upsilon) - L| d\tau\, d\upsilon \geqslant \varepsilon\Big\} \\ &\qquad \subseteq \Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}\Big|\Big\{(\tau,\upsilon)\in I_{t,s} : |f(\tau,\upsilon) - L| \geqslant \frac{\varepsilon}{2}\Big\}\Big| \geqslant \frac{\varepsilon}{2R}\Big\}. \end{aligned}$$

Since $f(\tau,\upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$, it follows that later set belongs to $\mathcal{I}_2$, which implies

$$\Big\{(t,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)} \iint_{(\tau,\upsilon)\in I_{t,s}} |f(\tau,\upsilon) - L| d\tau\, d\upsilon \geqslant \varepsilon\Big\} \in \mathcal{I}_2.$$

This demonstrates that $f(\tau,\upsilon) \xrightarrow{P} L(N_{\Theta_F}(\mathcal{I}_2))$. □

In the following, we investigate the relationship between $\mathcal{I}_2$-double statistical convergence and $\mathcal{I}_2$-lacunary double statistical convergence for two dimensional measurable functions.

THEOREM 2.2. *Let* $\Theta_F = \{g(t), h(s)\}$ *be a double lacunary function and* $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ *be a nontrivial ideal,* $f(\tau, \upsilon) \xrightarrow{P} L(S_F^2(\mathcal{I}_2))$ *implies* $f(\tau, \upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$ *if and only if* $\liminf_t \xi(t) > 1$ *and* $\liminf_s \varphi(s) > 1$. *If* $\liminf_t \xi(t) = 1$ *and* $\liminf_s \varphi(s) = 1$, *then there exists a bounded two dimensional function* $f(\tau, \upsilon)$ *which is* $\mathcal{I}_2$*-double statistically convergent but not* $\mathcal{I}_2$*-double lacunary statistically convergent.*

PROOF. Suppose $\liminf_t \xi(t) > 1$ and $\liminf_s \varphi(s) > 1$; then we can find $\psi > 0$ such that $1 + \psi \leqslant \xi(t)$ and $1 + \psi \leqslant \varphi(s)$ for sufficiently large t and s. This implies $\frac{\alpha(t)}{g(t)} \geqslant \frac{\psi}{1+\psi}$ and $\frac{\beta(s)}{h(s)} \geqslant \frac{\psi}{1+\psi}$. If $f(\tau, \upsilon) \xrightarrow{P} L(S_F^2(\mathcal{I}_2))$ then for every $\varepsilon > 0$, we obtain the following:

$$\begin{aligned}
\frac{1}{g(t,s)}&|\{\tau \leqslant g(t) \text{ and } \upsilon \leqslant h(s) : |f(\tau, \upsilon) - L| \geqslant \varepsilon\}| \\
&\geqslant \frac{1}{g(t,s)}|\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - L| \geqslant \varepsilon\}| \\
&= \frac{\alpha(t,s)}{g(t,s)}\frac{1}{\alpha(t,s)}|\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - L| \geqslant \varepsilon\}| \\
&\geqslant \Big(\frac{\psi}{1+\psi}\Big)^2\frac{1}{\alpha(t,s)}|\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - L| \geqslant \varepsilon\}|.
\end{aligned}$$

Then for any $\delta > 0$, we have

$$\begin{aligned}
\Big\{(t,s) \in \mathbb{N}\times\mathbb{N} : &\frac{1}{\alpha(t,s)}|(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - L| \geqslant \varepsilon| \geqslant \delta\Big\} \\
&\subseteq \Big\{(t,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g(t,s)}|\tau \leqslant g(t) \text{ and} \\
&\qquad \upsilon \leqslant h(s) : |f(\tau, \upsilon) - L| \geqslant \varepsilon| \geqslant \delta\Big(\frac{\psi}{1+\psi}\Big)^2\Big\} \in \mathcal{I}_2.
\end{aligned}$$

Therefore $f(\tau, \upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2)$ and this proves the sufficiency.

On the other side, assume that $\liminf_t \xi(t) = 1$ and $\liminf_s \varphi(s) = 1$. Let us choose a double subsequence function $g(\eta_i, \vartheta_j) = g(\eta_i) \cdot h(\vartheta_j)$ of the lacunary double function Θ_F such that

$$\begin{gathered}
\frac{g(\eta_i)}{g(\eta_i - 1)} < 1 + \frac{1}{i} \quad \text{and} \quad \frac{h(\vartheta_j)}{h(\vartheta_j - 1)} < 1 + \frac{1}{j}, \\
\frac{g(\eta_i - 1)}{g(\eta_{i-1})} > i \quad \text{and} \quad \frac{h(\vartheta_j - 1)}{h(\vartheta_{j-1})} > j
\end{gathered}$$

where $\eta_i \geqslant \eta_{i-1} + 2$, and $\vartheta_j \geqslant \vartheta_{j-1} + 2$.

Let us define $f(x, y)$ as follows:

$$f(x,y) = \begin{cases} 1, & \text{if } (x,y) \in I_{\eta_i,\vartheta_j} \\ 0, & \text{otherwise} \end{cases}.$$

Then, for any real L,

$$\frac{1}{\alpha(\eta_i,\vartheta_j)}\iint\limits_{(\tau,\upsilon)\in I_{\eta_i,\vartheta_j}}|f(x,y)-L|dx\,dy=|1-L|\quad\text{for } i,j=1,2,\dots$$

$$\frac{1}{\alpha(t,s)}\iint\limits_{(\tau,\upsilon)\in I_{t,s}}|f(x,y)-L|dx\,dy=|L|\quad\text{for } (t,s)\neq(\eta_i,\vartheta_j).$$

Then it is obvious that $f(\tau,\upsilon)$ does not belong to $N_{\Theta_F}(\mathcal{I}_2)$. Since $f(\tau,\upsilon)\in F(\ell)^2_\infty$, Theorem 2.1(3) implies that $f(\tau,\upsilon)\overset{P}{\nrightarrow}L(S_{\Theta_F}(\mathcal{I}_2))$.

If m and n are any sufficiently large integers we can find unique i and j for

$$g(\eta_i-1)\leqslant m\leqslant g(\eta_{i+1}-1)\quad\text{and}\quad h(\vartheta_j-1)\leqslant n\leqslant h(\vartheta_{j+1}-1).$$

Afterward,

$$\begin{aligned}
&\frac{\varepsilon}{mn}|\{\tau\leqslant m,\upsilon\leqslant n:|f(\tau,\upsilon)-L|\geqslant\varepsilon\}|\\
&\quad\leqslant\frac{1}{mn}\int_{x=1}^{m}\int_{y=1}^{n}|f(x,y)|dx\,dy\\
&\quad\leqslant\Big(\frac{g(\eta_i-1)+\alpha(\eta_i)}{g(\eta_i-1)}\Big)\cdot\Big(\frac{h(\vartheta_j-1)+\beta(\vartheta_j)}{h(\vartheta_j-1)}\Big)\\
&\quad\leqslant\frac{g(\eta_i-1,\vartheta_j-1)}{g(\eta_i-1,\vartheta_j-1)}+\frac{g(\eta_i-1)\cdot\beta(\vartheta_j)}{g(\eta_i-1)\cdot h(\vartheta_j-1)}\\
&\qquad+\frac{\alpha(\eta_i)\cdot h(\vartheta_j-1)}{g(\eta_i-1)\cdot h(\vartheta_j-1)}+\frac{\alpha(\eta_i)\cdot\beta(\vartheta_j)}{g(\eta_i-1)\cdot h(\vartheta_j-1)}\\
&\quad\leqslant 1+\frac{\beta(\vartheta_j)}{h(\vartheta_j-1)}+\frac{\alpha(\eta_i)}{g(\eta_i-1)}+\frac{\alpha(\eta_i)\cdot\beta(\vartheta_j)}{g(\eta_i-1)\cdot h(\vartheta_j-1)}\\
&\quad\leqslant 1+\frac{h(\vartheta_j)-h(\vartheta_j-1)}{h(\vartheta_j-1)}+\frac{g(\eta_i)-g(\eta_i-1)}{g(\eta_i-1)}\\
&\qquad+\frac{[g(\eta_i)-g(\eta_i-1)]\cdot[h(\vartheta_j)-h(\vartheta_j-1)]}{g(\eta_i-1)\cdot h(\vartheta_j-1)}\\
&\quad\leqslant 1+\frac{h(\vartheta_j)}{h(\vartheta_j-1)}-1+\frac{g(\eta_i)}{g(\eta_i-1)}-1\\
&\qquad+\Big(\frac{g(\eta_i)}{g(\eta_i-1)}-1\Big)\cdot\Big(\frac{h(\vartheta_j)}{h(\vartheta_j-1)}-1\Big)\\
&\quad\leqslant\Big(1+\frac{1}{i}\Big)+\Big(1+\frac{1}{j}\Big)+\frac{1}{ij}-1\\
&\quad\leqslant 1+\frac{1}{i}+\frac{1}{j}+\frac{1}{ij}\leqslant C
\end{aligned}$$

where C is any sufficient large constant. Hence $f(\tau,\upsilon)$ is $\mathcal{I}_2-$double statistically convergent for any nontrivial ideal $\mathcal{I}_2$. $\square$

For the next result we assume that the double lacunary function Θ_F satisfies the condition that for any set $\hat{G}_2 \in \mathcal{F}(\mathcal{I}_2)$,

$$\bigcup_{m,n}\{(m,n) : g(t-1) < m < g(t) \ \&\ h(s-1) < n < h(s), (t,s) \in \hat{G}_2\} \in \mathcal{F}(\mathcal{I}_2).$$

THEOREM 2.3. *Let* $\Theta_F = \{g(t), h(s)\}$ *be a double lacunary function and* $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ *be a nontrivial ideal,* $f(\tau,\upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$ *implies* $f(\tau,\upsilon) \xrightarrow{P} L(S_F^2(\mathcal{I}_2))$ *if and only if* $\limsup_t \xi(t) < \infty$ *and* $\limsup_s \varphi(s) < \infty$.

PROOF. Suppose that $\limsup_t \xi(t) < \infty$ and $\limsup_s \varphi(s) < \infty$. Then, there exist $0 < R < \infty$ and $0 < S < \infty$ such that $\xi(t) < R$ and $\varphi(s) < S$, for all $t \geqslant 1$ and $s \geqslant 1$. Suppose that $f(\tau,\upsilon) \xrightarrow{P} L(S_{\Theta_F}(\mathcal{I}_2))$ and for $\varepsilon, \delta, \delta^* > 0$ define the sets

$$\hat{G}_2 = \Big\{(t,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}|\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}| < \delta\Big\},$$

$$\hat{E}_2 = \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}|\{\tau \leqslant m, \upsilon \leqslant n : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}| < \delta^*\Big\}.$$

It is clear from our assumption that $\hat{G}_2 \in \mathcal{F}(\mathcal{I}_2)$, the filter associated with the ideal $\mathcal{I}_2$. Additionally, we observe that

$$\tilde{A}_{i,j} = \frac{1}{\alpha(i,j)}|\{(\tau,\upsilon) \in I_{i,j} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}| < \delta,$$

for all $(i,j) \in \hat{G}_2$. Let $(m,n) \in \mathbb{N}\times\mathbb{N}$ be such that $g(t-1) < m < g(t)$ and $h(s-1) < n < h(s)$ for all $(t,s) \in \hat{G}_2$. Moreover, $\alpha(t,s) = \alpha(t)\cdot\beta(s) = [g(t) - g(t-1)]\cdot[h(s) - h(s-1)] \leqslant g(t,s) - g(t-1,s)$, and $g(1) \leqslant h(1) \leqslant g(2) \leqslant h(2) \leqslant \cdots \leqslant g(r-1) \leqslant h(r-1) \leqslant g(r) \leqslant h(r)$ as $r \to \infty$, we obtain

$$\begin{aligned}
&\frac{1}{mn}|\{\tau \leqslant m, \upsilon \leqslant n : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}|\\
&\quad= \frac{1}{g(t-1,s-1)}|\{\tau \leqslant g(t), \upsilon \leqslant h(s) : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}|\\
&\quad= \frac{1}{g(t-1,s-1)}|\{(\tau,\upsilon) \in I_{2,2} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}| + \cdots +\\
&\qquad + \frac{1}{g(t-1,s-1)}|\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}|\\
&\quad\leqslant \frac{g(2,2) - g(2,1)}{g(t-1,s-1)}\frac{1}{\alpha(2,2)}|\{(\tau,\upsilon) \in I_{2,2} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}|\\
&\qquad + \frac{g(3,3) - g(2,3)}{g(t-1,s-1)}\frac{1}{\alpha(3,3)}|\{(\tau,\upsilon) \in I_{3,3} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}|\\
&\qquad + \frac{g(t,s) - g(t-1,s)}{g(t-1,s-1)}\frac{1}{\alpha(t,s)}|\{(\tau,\upsilon) \in I_{t,s} : |f(\tau,\upsilon) - L| \geqslant \varepsilon\}|\\
&\quad\leqslant \frac{g(2,2) - g(2,1)}{g(t-1,s-1)}\tilde{A}_{2,2} + \frac{g(3,3) - g(2,3)}{g(t-1,s-1)}\tilde{A}_{3,3} + \cdots + \frac{g(t,s) - g(t-1,s)}{g(t-1,s-1)}\tilde{A}_{t,s}\\
&\quad\leqslant \sup_{(i,j)\in\hat{G}_2} \tilde{A}_{i,j}\frac{g(t,s)}{g(t-1,s-1)} < RS\delta.
\end{aligned}$$

Choosing $\delta^* = \frac{\delta}{RS}$ and in view of the fact that

$$\bigcup_{m,n}\{(m,n) : g(t-1) < m < g(t) \ \&\ h(s-1) < n < h(s), (t,s) \in \hat{G}_2\} \subset \hat{E}_2,$$

where $\hat{G}_2 \in \mathcal{F}(\mathcal{I}_2)$ it follows from our assumption on Θ_F that the set $\hat{E}_2 \in \mathcal{F}(\mathcal{I}_2)$ and this completes the proof of the theorem. □

THEOREM 2.4. *The set $S_{\Theta_F}(\mathcal{I}_2) \cap F(\ell)^2_\infty$ is a closed subset of $F(\ell)^2_\infty$, where as usual $F(\ell)^2_\infty$ is Banach space of all bounded real functions endowed with the supremum norm.*

PROOF. Suppose that $f_{m,n} = f_{m,n}(\tau, \upsilon) \in S_{\Theta_F}(\mathcal{I}_2) \cap F(\ell)^2_\infty$ is a P-convergence function and converges to $f(\tau,\upsilon) \in F(\ell)^2_\infty$. Since $f_{m,n} \in S_{\Theta_F}(\mathcal{I}_2)$, there exists $\mu_{m,n}$ for $m = 1,2,3,\dots$ and $n = 1,2,3,\dots$ such that $S_{\Theta_F}(\mathcal{I}_2) - P - \lim f_{m,n}(\tau,\upsilon) = \mu$. We first show that the sequence $\mu_{m,n}$ is $P-$convergent to some number μ and $f = f(\tau,\upsilon)$, which is a real valued function of two variables measurable on $(1,\infty) \times (1,\infty)$, is $\mathcal{I}_2$-double lacunary statistically convergent to μ. Since $f_{m,n}(\tau,\upsilon) \to \mu_{m,n}(S_{\Theta_F}(\mathcal{I}_2))$. As $f_{m,n} \to f$ implies $f_{m,n}$ is a multidimensional Cauchy function. Therefore for every $\varepsilon > 0$, there exists a positive integer n_0 such that for every $p \geqslant m \geqslant n_0$ and $q \geqslant n \geqslant n_0$, we obtain $|f_{p,q} - f_{m,n}| < \frac{\varepsilon}{3}$. Since $f_{m,n}(\tau,\upsilon) \xrightarrow{P} \mu_{m,n}(S_{\Theta_F}(\mathcal{I}_2))$, so for each $\varepsilon > 0$ and $\tilde{\delta} > 0$, if we denote the sets

$$R_1 = \Big\{(t,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}\Big|\Big\{(\tau,\upsilon) \in I_{t,s} : |f_{m,n}(\tau,\upsilon) - \mu_{m,n}| \geqslant \frac{\varepsilon}{3}\Big\}\Big| < \frac{\tilde{\delta}}{3}\Big\},$$
$$R_2 = \Big\{(t,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\alpha(t,s)}\Big|\Big\{(\tau,\upsilon) \in I_{t,s} : |f_{p,q}(\tau,\upsilon) - \mu_{p,q}| \geqslant \frac{\varepsilon}{3}\Big\}\Big| < \frac{\tilde{\delta}}{3}\Big\},$$

then $\emptyset \neq R_1 \cap R_2 \in \mathcal{F}(\mathcal{I}_2)$. Let $(t,s) \in R_1 \cap R_2$, then we obtain

$$\frac{1}{\alpha(t,s)}\Big|\Big\{(m,n) \in I_{t,s} : |f_{m,n}(\tau,\upsilon) - \mu_{m,n}| \geqslant \frac{\varepsilon}{3}\Big\}\Big| < \frac{\tilde{\delta}}{3},$$
$$\frac{1}{\alpha(t,s)}\Big|\Big\{(m,n) \in I_{t,s} : |f_{p,q}(\tau,\upsilon) - \mu_{p,q}| \geqslant \frac{\varepsilon}{3}\Big\}\Big| < \frac{\tilde{\delta}}{3},$$

which implies that

$$\frac{1}{\alpha(t,s)}\Big|\Big\{(m,n) \in I_{t,s} : |f_{m,n}(\tau,\upsilon) - \mu_{m,n}| \geqslant \frac{\varepsilon}{3} \vee |f_{p,q}(\tau,\upsilon) - \mu_{p,q}| \geqslant \frac{\varepsilon}{3}\Big\}\Big| < \tilde{\delta} < 1.$$

This shows that there exists a pair $(\tau_0,\upsilon_0) \in I_{t,s}$ for which $|f_{m,n}(\tau_0,\upsilon_0) - \mu_{m,n}| < \frac{\varepsilon}{3}$ and $|f_{p,q}(\tau_0,\upsilon_0) - \mu_{p,q}| < \frac{\varepsilon}{3}$. Moreover, for $p \geqslant m \geqslant n_0$ and $q \geqslant n \geqslant n_0$, we get

$$\begin{aligned}|\mu_{m,n} - \mu_{p,q}| = |\mu_{p,q} - f_{p,q}(\tau_0,\upsilon_0)| &+ |f_{p,q}(\tau_0,\upsilon_0) - f_{m,n}(\tau_0,\upsilon_0)| \\ &+ |f_{m,n}(\tau_0,\upsilon_0) - \mu_{m,n}| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Hence $(\mu_{m,n})$ is a Cauchy double sequence in $\mathbb{R}$ (or $\mathbb{C}$) and consequently there is a number μ such that $\mu_{m,n} \xrightarrow{P} \mu$. Now to prove the theorem it is sufficient to show that the real valued measurable function of two variables $f = f(\tau,\upsilon) \to \mu(S_{\Theta_F}(\mathcal{I}_2))$.

Since $f_{m,n} = f_{m,n}(\tau, \upsilon) \in S_{\Theta_F}(\mathcal{I}_2) \cap F(\ell)^2_\infty$ is a P-convergent function and P-convergence to $f(\tau, \upsilon) \in F(\ell)^2_\infty$. Therefore, for each $\varepsilon > 0$, there exists a positive integer $n_1(\varepsilon)$ such that

$$|f_{m,n}(\tau, \upsilon) - f_{m,n}| < \frac{\varepsilon}{3} \quad \text{for} \quad m, n \geqslant n_1(\varepsilon).$$

Also $\mu_{m,n} \xrightarrow{P} \mu$, so for each $\varepsilon > 0$, we can find another positive integer $n_2(\varepsilon)$ such that

$$|\mu_{m,n} - \mu| < \frac{\varepsilon}{3}, \quad \forall m, n \geqslant n_2(\varepsilon).$$

Choose $n_3(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$ and $m_0, n_0 \geqslant n_3(\varepsilon)$. Then for any $(\tau, \upsilon) \in \mathbb{N} \times \mathbb{N}$

$$\begin{aligned} |f(\tau, \upsilon) - \mu| &\leqslant |f(\tau, \upsilon) - f_{m_0,n_0}(\tau, \upsilon)| + |f_{m_0,n_0}(\tau, \upsilon) - \mu_{m_0,n_0}| + |\mu_{m_0,n_0} - \mu| \\ &< \frac{\varepsilon}{3} + |f_{m_0,n_0}(\tau, \upsilon) - \mu_{m_0,n_0}| + \frac{\varepsilon}{3}, \end{aligned}$$

and therefore the containment

$$\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - \mu| \geqslant \varepsilon\} \subseteq \Big\{(\tau, \upsilon) \in I_{t,s} : |f_{m_0,n_0}(\tau, \upsilon) - \mu_{m_0,n_0}| \geqslant \frac{\varepsilon}{3}\Big\}$$

implies

$$\begin{aligned} &\frac{1}{\alpha(t,s)}|\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon)| \geqslant \varepsilon\}| \\ &\qquad\qquad \leqslant \frac{1}{\alpha(t,s)}|\{(\tau, \upsilon) \in I_{t,s} : |f_{m_0,n_0}(\tau, \upsilon) - \mu_{m_0,n_0}| \geqslant \varepsilon\}. \end{aligned}$$

In addition, for any $\tilde{\delta} > 0$ we obtain

$$\begin{aligned} &\Big\{(t,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\alpha(t,s)}\Big|\Big\{(\tau, \upsilon) \in I_{t,s} : |f_{m_0,n_0}(\tau, \upsilon) - \mu_{m_0,n_0}| \geqslant \frac{\varepsilon}{3}\Big\}\Big| < \tilde{\delta}\Big\} \\ &\qquad \subseteq \Big\{(t,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\alpha(t,s)}\Big|\Big\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - \mu| \geqslant \varepsilon\Big\}\Big| < \tilde{\delta}\Big\}. \end{aligned}$$

Because

$$\Big\{(t,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\alpha(t,s)}\Big|\Big\{(\tau, \upsilon) \in I_{t,s} : |f_{m_0,n_0}(\tau, \upsilon) - \mu_{m_0,n_0}| \geqslant \frac{\varepsilon}{3}\Big\}\Big| < \tilde{\delta}\Big\} \in \mathcal{F}(\mathcal{I}_2).$$

Therefore

$$\Big\{(t,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\alpha(t,s)}|\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - \mu| \geqslant \varepsilon\}| < \tilde{\delta}\Big\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence,

$$\Big\{(t,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\alpha(t,s)}|\{(\tau, \upsilon) \in I_{t,s} : |f(\tau, \upsilon) - \mu| \geqslant \varepsilon\}| < \tilde{\delta}\Big\} \in \mathcal{I}_2.$$

This demonstrates that $f = f(\tau, \upsilon) \xrightarrow{P} \mu(S_{\Theta_F}(\mathcal{I}_2))$. $\square$

COROLLARY 2.1. *The set* $(S^2_F(\mathcal{I}_2)) \cap F(\ell)^2_\infty$ *is a closed subset of* $F(\ell)^2_\infty$.

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