Özkan Ergene, Ahmet Şükrü Özdemir

A STUDY ON THE PRE-SERVICE ELEMENTARY MATHEMATICS TEACHERS' KNOWLEDGE ON THE CONVERGENCE AND DIVERGENCE OF SERIES IN THE CONTEXT OF THEORY AND APPLICATION

RESUMEN

El foco de esta investigación es el examen del conocimiento teórico y práctico sobre la convergencia y la divergencia de la serie. En línea con el enfoque de investigación, los maestros pre-servicio de matemática sobre la convergencia y divergencia de la serie con la ayuda del problema de la vida real en el contexto de la teoría y la aplicación utilizando el concepto de series armónicas. La investigación se guió de un método cualitativo, estudio de caso. Los datos de la investigación han consistido en dos preguntas escritas y un problema y cuatro preguntas de entrevista que fueron formadas por los investigadores y se aseguró su validez y confiabilidad. El estudio llegó a la conclusión de que maestros de pre-servicio matemática tienen conocimientos teoríco sobre la convergencia de series y series armónicas, su percepción de las series, las series armónicas, los conceptos de convergencia y divergencia cambiaron en el proceso de solicitud y adoptaron diferentes enfoques en la resolución de problemas.

PALABRAS CLAVE:

- *Enseñanza del cálculo*
- *Series armónicas*
- *Convergencia y divergencia*
- *Teoría y aplicación*

ABSTRACT

The focus of this research is the examination of the theoretical and practical knowledge regarding the convergence and divergence of the series. In line with the research focus, it is aimed to reveal the approaches of pre-service elementary mathematics teachers on the convergence and divergence of the series with the help of the real-life problem in the context of theory and application by using the concept of harmonic series. The research carried out the qualitative research method is designed according to the case study. The data of the research have consisted of two written questions and one problem and four interview questions which were formed by the researchers and their validity and reliability were ensured. The study concluded that pre-service teachers have theoretical knowledge on the convergence of series and harmonic series. Also, their perception of series, harmonic series, convergence and divergence concepts changed in the application process and they performed different approaches in problem-solving.

KEYWORDS:

- *Calculus Teaching*
- *Harmonic Series*
- *Convergence and Divergence*
- *Theoretical and Application*

Revista Latinoamericana de Investigación en Matemática Educativa (2020) *23* (2): 203 -232.
Recepción: Diciembre 9, 2019 / Aceptación: Julio 12, 2020. DOI: 10.12802/relime.20.2323

RESUMO

O foco desta investigação é a análise do conhecimento teórico / prático sobre a convergência e divergência de uma série. Em consonância com o foco da pesquisa, pretende-se analisar as abordagens dos professores estagiários de matemática sobre a convergência e divergência da série com a ajuda de problemas da vida real no contexto da teoria e da aplicação utilizando-se o conceito de série harmónica. A pesquisa foi realizada usando o método de pesquisa qualitativa e foi projetada de acordo com o estudo de caso. Os dados da pesquisa consistiram em duas questões escritas e um problema, e uma entrevista com quatro perguntas formadas pelos investigadores, sendo que a sua validade e fiabilidade foram garantidas. A investigação concluiu que os professores estagiários têm conhecimento suficiente sobre a convergência de séries e séries harmónicas. Além disso, a sua percepção de séries e séries harmônicas, conceitos de convergência e divergência mudaram durante o processo de aplicação, realizando diferentes abordagens na resolução dos problemas.

PALAVRAS CHAVE:

- *Ensino de Cálculo*
- *Série Harmônica*
- *Convergência e Divergência*
- *Teórica e Aplicação*

RÉSUMÉ

L'objectif de cette recherche consiste à étudier les connaissances théoriques et pratiques relatives à la convergence et à la divergence des séries. Conformément à cet objectif, l'étude vise à révéler les approches des futurs enseignants de mathématiques au secondaire liées à la convergence et à la divergence de la série harmonique en théorie et en pratique à l'aide d'un problème de la vie réelle. La méthode de recherche adoptée relève d'une démarche qualitative dans le cadre d'une étude de cas. Les données de la recherche comprennent d'une part, deux questions et un problème écrit et d'autre part, quatre questions d'entretien dont la validité et la fiabilité sont assurées par le chercheur lui-même. Les résultats de l'étude montrent que les futurs enseignants atteignent un niveau de connaissances théoriques sur les séries et sur la convergence des séries harmoniques. Par ailleurs, l'étude révèle aussi que les futurs enseignants se diffèrent quant à l'attribution de sens aux concepts de série, de série harmonique, de divergence et d'infini, ainsi qu'à l'approche de la résolution de problèmes.

MOTS CLÉS:

- *Enseignement de l'Analyse*
- *Série Harmonique*
- *Convergence et Divergence*
- *Théorique et Pratique*

1. Introduction

The concept of series is closely related to the teaching of concepts in mathematics such as sequences, limits, derivatives and integrals. Indeed, series are utilized in the calculation of Riemann integral, which is used to calculate definite integrals, in the calculation of differential equations, in the conversion of repeating decimals, and in the expansion of functions (Taylor Series). In addition to mathematics, series are used in science, specifically in biology, in the creation of population distribution models, and in economy to calculate interest rates of bank accounts. Thus, series do not only contribute to the improvement of Calculus courses, but they also figure in the application of various disciplines of science and mathematics. Moreover, they are incorporated in the high school curriculum and university level mathematics and Calculus courses in many countries (González-Martín, 2013a).

The teaching of series firstly includes finding the rules of number patterns and sequences, creating formulae that correspond to the discovered rules, and finding the general term of the sequences. Then, the sums of the series, whose general terms are known, are calculated. The aim of the calculation is to determine the sum of the series. That is, if the sum of the series exists as a real number, it is called a convergent series and if the sum does not exist as a real number, it is called a divergent series. Indeed, these calculations are valid for series that are defined within the set of real number. On the other hand, calculations for series that are defined within the set of complex numbers are much different.

There are two principle methods to determine the convergence or the divergence (character) of a series. The first method is to define the convergence of a_n sequence of partial sums comprising the first n term of the series. The second method is determining convergence by using various criteria such as d'Alembert's ratio test, Cauchy's Root Test and Raabe's Test. Both methods to define the character of a series consist several mathematical concepts, including function, sequence, limit, integral and infinity. For instance, the concepts of limit and integral are used in the application of a certain solution or a test to define the character of the series and the calculations are based on these concepts. On the other hand, there are also special forms of series such as the Taylor series produced by the patterns of derivatives of a function (Martin, 2012).

Another special form of series is the harmonic series denoted by $(a_n)=\sum_{n=1}^{\infty}\frac{1}{n}$. Harmonic series is among the generic examples for series whose general terms tend to zero, while the series itself diverges. Harmonic series which are also utilised in music theory (Aliyev & Dil, 2012) as well as in mathematical disciplines such as calculus and number theory are closely followed by mathematicians and has been studied for almost 400 years since the era of Euler. The research has produced multiple proofs of divergence of harmonic series. One of the reasons of why harmonic series is followed closely includes the scenarios where the $\sum_{n=1}^{\infty}\frac{1}{n^p}$ Riemann series defined for $p>0$ becomes a special case if $p=1$, and where the $1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$ proposition with continuously decreasing sum diverges to infinity.

Series is considered to have a complex structure (González-Martín, Nardi & Biza, 2011) and a difficult nature (Cornu, 1991; González-Martín, 2013b) because it is related to many subjects of calculus, it contains interesting elements such as harmonic series and it is utilised in various disciplines, including science. Literature review on mathematics education shows that the number of research studies focused on series is less than the number of research studies on derivative, limit and other such concepts of calculus. The small number of studies on series in the literature is striking.

Gonzàlez-Martin, Nardi and Biza (2011) indicated that the studies in literature which focus on series were examined in three groups consisting of the definition of the concept, student experiences in the learning of the concept, and the suggestions on the teaching of the concept. Additionally, there exist studies which examined series together with related concepts such as integral (Bezuidenhout & Olivier, 2000), derivative and sequences, as well as other studies which examined its relations with its inherent concepts such as convergence (Boschet, 1983; Robert, 1982) and infinity (Fischbein, 1987). There are also studies focused on special forms of series that include Taylor Series (Martin, 2012) and Power Series (Kung & Speer, 2010). Participants in the conducted studies varied from high school and undergraduate students to teachers and lecturers.

Boschet (1983) and Robert (1982) highlighted the lack of practice and the little use of graphic representations in the teaching of series. It is asserted that in addition to use of graphic representations in the teaching of series, using visual reasoning could also improve the conceptual understanding of students (Alcock & Simpson, 2004; Bagni, 2000; Alcock & Simpson, 2004; Bagni; 2000; Fisher, 2016). Furthermore, it is acknowledged that the traditional teaching of series disregards the procedural and conceptual distinction.

In the structuring the idea of infinity in series, the fact that infinite terms cannot be perceived as a whole (Fischbein, 1987) and the perception that infinite terms imply infinite sum (Bagni, 2000; 2008) lead to difficulties in student understanding of the concept of series (Bagni, 2005). Furthermore, it is underlined that students think that symbolic representations of series are different (Kidron, 2002). Similarly, Mamona (1990) emphasised confusion of students regarding sequences and their incapacity to consider series as functions.

According to literature review, it is evident that students experience various difficulties in the comprehension of series. One of the main reasons of these difficulties is students' lack of conceptual understanding of series (Fisher, 2016; Alcock & Simpson, 2004). The constructivist approach adopted in recent years emphasizes that students should be able to apply their academic knowledge to real-life and should comprehend concepts on a more functional level. Thus, it is necessary to elaborate on the fields of application of various mathematical concepts and to evaluate theoretical knowledge of students along with their knowledge of application (Bransford, Brown & Cocking, 2000). It is within this context that series should be associated with real-life situations and problems and real-life situations should be used in teaching and evaluation processes of series.

In the light of these situations, the focus of this study consists of "examination of the knowledge of pre-service elementary mathematics teachers' on the convergence and the divergence of series in the context of theoretical knowledge and practical knowledge". In accordance with the study focus, the purpose is to investigate the approach of pre-service teachers to the convergence and the divergence of series using the concept of harmonic series in the context of theoretical knowledge and practical knowledge with the support of a real-life problem. For this purpose, following research questions were formulated:

i. What is the theoretical knowledge of pre-service elementary mathematics teachers on the convergence and the divergence of series?
ii. What is the theoretical knowledge of pre-service elementary mathematics teachers on the convergence and the divergence of harmonic series?
iii. How do the pre-service elementary mathematics teachers apply their knowledge on the convergence and the divergence of series to the process of solving a real-life problem?

1.1. *Conceptual Framework and Significance of the Study*

The Calculus course comprises abstract, complex and hierarchical concepts (Nesbit, 1996). Therefore, students encounter with difficulties in the learning and teaching of these concepts (Ergene & Özdemir, 2020; Cetin, Dane & Bekdemir, 2012). Particularly, series is an important concept in order to understand other concepts of Calculus and it is strongly related to sequences. Research studies which were conducted about the convergence of sequences mostly focused on developing methods to teach the concept but there are limited number of studies that examine pre-service teachers' content knowledge about this concept (Alcock & Simpson, 2004). Similarly, Kung and Speer (2013) emphasized that even though series is an important part of calculus courses, little is known about how students think about this concept. In addition, not only students from all levels but also teachers have a difficulty in understanding the concept of series (Lindaman & Gay, 2012). Thus, it was suggested that using specially designed problems and questions about this concept enable students to improve their content knowledge (Przenioslo, 2005).

Contextualized mathematical problems can be used in order to promote application of mathematical knowledge flexibly and adaptively. Research have shown that presenting mathematical tasks enriched with real life contexts that are meaningful for students can help them internalize problem solving experiences (Stillman, 2000) because students shape meaningful tasks themselves by establishing relations with them (Busse, 2005). Using real contexts not only enable students to use different strategies but also facilitate their comprehension by activating knowledge structures (Ceci & Roazzi, 1993). Therefore, there is a need to make explicit connection between theory and practice, i.e. between mathematical content and the real world in order to make the mathematical content relevant to students.

A framework helps conceptualize and design research studies by providing a structure. Particularly, a research framework shapes formulation of the research questions for which the answers are sought and how the research concepts, constructs and processes are defined (Lester, 2005). In the present study, knowledge on the convergence and divergence of series were examined in terms of both theoretical and practical knowledge as a conceptual framework. The knowledge that can be obtained from textbooks consisting of series, harmonic series, mathematical definitions and notations was considered as theoretical knowledge. On the other hand, the use of this knowledge in solving real-life problems has been determined as practical knowledge. That is, "theoretical knowledge" refers to the knowledge in the textbooks, while practical knowledge refers to the "use" of the theoretical in a "contextual" situation."

The present study was designed, therefore, to investigate the transition between knowledge and context on students' interpretation of series. The findings may have both theoretical and practical implications. From a theoretical perspective, investigating the role of real-life problem context on understanding series is likely to allow inference about the nature of knowledge. In addition, from a practical perspective, understanding how pre-service teachers function in the given context may contribute to the design and manipulation of instructional conditions. Since there are limited number of studies which are conducted about series especially harmonic series, we believe that the findings of the study will contribute to mathematics education literature. Furthermore, the findings of the study may also point the relationship between series and infinity / limit concepts. Moreover, the findings of the study can enlighten teacher educators about teacher education programs and the content of the mathematics courses especially Calculus courses.

2. Methodology

The qualitative research method was adopted in this study in order to reveal pre-service teachers' approaches to the convergence and divergence of series in the context of theoretical and practical knowledge using the concept of harmonic series.

2.1. *Research Design*

In this study, case study was utilized which is one of the qualitative research methods. In a case study, it is required to clearly define the subject matter of the study (Yin, 1994). Theoretical and practical knowledge of pre-service teachers on series have been considered as individual cases to be studied within the scope of the study.

2.2. *Participants*

The participants of this study consist of 50 pre-service teachers who were studying at the Elementary Mathematics Education program of one of the public universities located in Marmara Region of Turkey and who all have completed Calculus I-II-III courses successfully at the time of the study. The participants of this study have been selected by using purposive sampling method (Patton, 1990) among the non-random sampling methods.

2.3. *Context of the Study*

The context for this study was an Elementary Mathematics Education program at one of the public universities in Turkey. This program is four-year undergraduate program designed in order to train pre-service elementary mathematics teachers to teach mathematics at grade levels 5 to 8 (ages 10–14). Calculus I and Calculus II courses are offered in the second year of the program while Calculus III course is offered in the third year of the program. In addition to Calculus courses, other mathematics content courses such as Statistics, Probability, Algebra are offered. In the last two years of the program, pre-service teachers take courses related to methods of teaching mathematics and teaching practice.

In the present study, all pre-service teachers selected as participants studied sequences, series and harmonic series in Calculus III course and sequences in Calculus I course and completed all these courses successfully. They studied series for 24 class hours in 8 weeks in Calculus III course and studied sequences for a total of 18 class hours, of which 12 class hours in 2 weeks were allocated for Calculus I course and 6 class hours in 2 weeks were allocated for Calculus III course.

2.4. *Data Collection Tools*

In this study, questionnaire and semi-structured interview protocol were used as data collection tools. In the questionnaire, two open ended questions (Q_1 and Q_2) and one real-life problem (P_1) were presented to pre-service teachers in order to examine their knowledge on harmonic series and to reveal the discrepancies between theoretical and practical aspects of their knowledge (see Appendix I). In addition, semi-structured interviews were conducted with four pre-service teachers. During the interviews, four questions (I_1, I_2, I_3 and I_4) were asked pre-service teachers in order to further explore their responses to the questions and problem in the questionnaire to fully understand the reasons of their given responses (Table I).

TABLE I

Questions and the problem addressed in the study and their purposes

Code	*Question*	*Purpose*
Q_1	What does it mean when a series is convergent or divergent? Please explain.	To examine theoretical knowledge of pre-service teachers on convergence and divergence
Q_2	Examine the convergence of $(a_n)=\sum_{n=1}^{\infty}\frac{1}{n}$ harmonic series. (Aliyev & Dil, 2012)	To examine theoretical knowledge of pre-service teachers on convergence and divergence of harmonic series

P_1	A water tank of 38 m^3 capacity is to be filled up; – $1\,m^3$ on the first day, – $\frac{1}{2}m^3$ on the second day, – $\frac{1}{3}m^3$ on the third day, – $\frac{1}{n}m^3$ on the nth day … Can the water tank be completely filled up in this manner? Please explain your answer.	To reveal application of pre-service teachers' knowledge on convergence and divergence to real-life situations
I_1	What are the differences between the convergence and the divergence of a series?	To obtain further information on the concepts of convergence and divergence
I_2	How much water do you estimate will be collected in the water tank? Can you express the amount of collected water in terms of quantitative data (mathematical expressions)?	To examine the concept of infinity and the ability to associate real numbers with the concept of infinity
I_3	Could you please repeat your solution to the presented question?	To obtain detailed information on the utilised representations
I_4	A water tank of 38 m^3 capacity is to be filled up; – $\frac{1}{2}m^3$ on the first day, – $\frac{1}{4}m^3$ on the second day, – $\frac{1}{8}m^3$ on the third day, – $\frac{1}{2^n}m^3$ on the nth day … Can the water tank be completely filled up in this manner? Please explain your answer.	To reveal ability of pre-service teachers to find the solution of a convergent series; and the ability of them to apply to a real-life situation; and to compare the answer with the answers given to P_1

Following the literature review, questions Q_1 and Q_2 addressed in the study were developed by the researchers in order to explore the theoretical knowledge of pre-service teachers on the convergence and the divergence of series. Similarly, problem P_1 and the interview question I_4 presented in the study were also developed by the researchers in order to explore the practical knowledge of pre-service teachers on the concepts of convergence and divergence by using the series. The studies of Yalçınkaya (2015) and González-Martín, (2013b) inspired the researchers in the design of problem P_1. Likewise, all the other questions presented in the interviews were also developed by the researchers. To take expert opinion, three mathematics education researchers were consulted for the questions after the purpose of the study and research questions were presented to them. These researchers have conducted research studies on the concepts of Calculus and they were instructors of Calculus courses in different universities at the time of the study. According to the expert opinions, the question I_3 was included in the study and revisions such as the statement "Please explain your answer" were made. After the revisions and additions, the questions and the problem were considered as appropriate to present.

2.5. *Data Analysis*

Data in this study were analysed by descriptive analysis method (Robson, 1993) in accordance with purpose of the study and research questions. The analysis of questions and answers have been completed through the descriptive analysis method in three phases consisting convergence and divergence of series, convergence and divergence of harmonic series, and evaluation of the real-life problem. During data analysis process, one mathematics education expert has also been consulted.

2.6. *The Role of the Researchers*

As qualitative research is inherently based on observation and interpretation, the role of researchers is essential in this process. The observations and experience of researchers were influential on the selection of the participants of the study. The study is structured on the experiences of the first researcher during his undergraduate education and on his belief that he felt inadequate in application of the concepts of series and harmonic series to real-life during his undergraduate education as well as the second researchers' experiences, opinions and guidance on the matter. Moreover, the time spent together with the participants in the courses and extracurricular environments and the various comments on series during natural and short conversations made us think that various approaches about the series can be evaluated.

2.7. *Validity and Reliability of the Study*

In qualitative research, "credibility" is used for internal validity and "transferability" is used for external validity (Lincoln & Guba, 1985). In order to ensure credibility in the study, data triangulation and investigator triangulation were preferred during the development of data collection tools and the data analysis process. For data triangulation, multiple sources of data consisting of written questions (Q_1 and Q_2) and the problem (P_1) and the interview questions (I_1, I_2, I_3 and I_4) were used. For investigator triangulation, more than one researcher involved in the data collection and data analysis processes. Participants' responses were analysed by the researchers individually and discrepancies were discussed and resolved. Since this is a qualitative study, it does not hold concern for generalization, but the findings can be transferred to similar contexts or settings. For this reason, the case addressed in the study was defined in detail for its transferability to similar contexts or settings. For dependability, audit trail was used. That is, the research steps taken from the start of the research to the development and reporting of the findings were described transparently. The fact that the study method and the process have been reported in detail, that the interview questions have been formulated in accordance with data obtained from the questions and problem addressed in the study, and that opinions of mathematics education experts have been consulted during the study process are evidences to the dependability of the study.

3. Findings

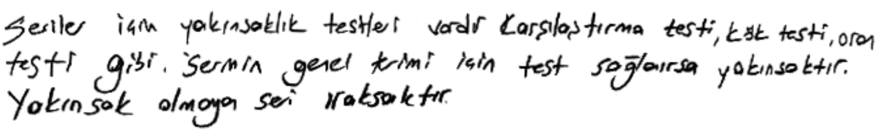
The findings of the study will be presented in the context of convergence and divergence of the series, convergence and divergence of harmonic series and the real-life problem, and they will be supported with the findings of interviews. In the tables, abbreviations were used for both answers of questions (see Appendix 1) and interviewed pre-service teachers. For the first question (Q_1) six different answers (A_{11}, A_{12}, …, A_{16}), for the second question (Q_2) five different answers (A_{21}, A_{22}, …, A_{25}) and for the real-life problem (P_1) three different categories (P_{11}, P_{12}, P_{33}) given by the pre-service teachers were determined. Also, interviewed pre-service teachers were coded as PST_1, PST_2, PST_3 and PST_4 according to their responses to Q_1, Q_2 and P_1 (Table II). For instance, the pre-service teacher who asserted that the harmonic series is convergent and that the water tank can be completely filled up was coded as PST_1. On the other hand, the pre-service teacher who stated that harmonic series is divergent and did not specify on the filling of the water tank was coded as PST_4.

TABLE II
Characteristics of interviewed pre-service teachers

Code	*Harmonic Series*		*Water Tank*		
	Convergent	*Divergent*	*Fillable*	*Unfillable*	*Not Specified*
PST_1	✓		✓		
PST_2	✓			✓	
PST_3		✓		✓	
PST_4		✓			✓

3.1. *Findings on Convergence and Divergence of the Series*

While almost all pre-service teachers (n=48) explained "the convergence of the series" as definition and expression, none of them explained "the divergence of the series". They referred to the divergence of the series as an opposite of the concept of convergence; that is, they used it conversely as "a series which is not convergent, is divergent", or "a series which is convergent, is not divergent" (Figure 1).


Seriler için yakınsaklık testleri vardır Karşılaştırma testi, kök testi, oran testi gibi. Serinin genel terimi için test sağlarsa yakınsaktır. Yakınsak olmayan seri ıraksaktır.

Translation of handwriting – "There are convergence tests for series. For instance, comparison test, root test, and ratio test. If the test is ensured for the general term of the series, then the series is convergent. A series that is not convergent, is divergent."

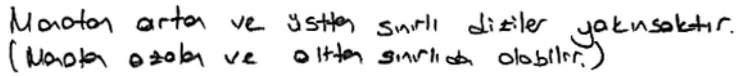
Figure 1. A response about convergence and divergence of the series

Pre-service teachers explained convergence of a series in various ways including the sequence of partial sums (n=15), Cauchy's Convergence Criterion (n=5), convergence criteria (n=4), verbal expressions (n=8), and phrases such as "the sum of the series tends to infinity" and "it is continuously increasing" (n=7) (Table III).

TABLE III
Frequency distribution of answers of the
pre-service teachers who explained convergence of a series

	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
n	15	5	4	8	7	11

Some of the pre-service teachers, on the other hand, explained the convergence of sequences rather than the convergence of series (Figure 2).



Translation of handwriting – "If a sequence is monotonically increasing and bounded from above, it is convergent (It may be monotonically decreasing and bounded from below as well)

Figure 2. A response about convergence of sequences rather than convergence of series

During the interviews, the pre-service teachers attempted to explain the differences between convergence and divergence of a series by providing definitions. Following excerpt illustrates this situation:

I (researcher): What are the differences between convergence of a series and divergence of a series?

PST_2: Convergence of a series means that each of almost all terms of the series tends to a number, while we could say that the divergence of a series means that none of the terms tends to a number.

I: So, is tending to the number significant for being convergence / divergence?

PST_2: Of course. But this number needs to be a real number. Besides, there are some criteria that we discussed in the course, such as the ratio test and the Raabe test. If it meets these criteria, then it is convergent. If it does not, then it is divergent.

All of the interviewed pre-service teachers claimed that the limit of a convergent series needs to be equal to a real number.

PST_4: For a series to be convergent, the limit of its general term must be equal to a real number. Although this is necessary, it is not sufficient. It also needs to satisfy at least one of the convergence criteria, for instance…

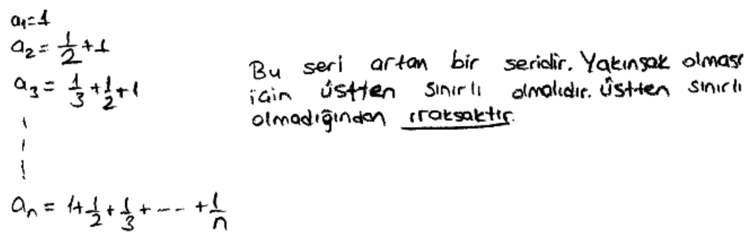
3.2. *Findings on Convergence and Divergence of Harmonic Series*

While most of the pre-service teachers (n=38) asserted that harmonic series is divergent, others (n=11) claimed that the harmonic series is convergent. A very small number of the pre-service teachers (n=2) stated nothing about the divergence or the convergence of the harmonic series (Table IV).

TABLE IV
Frequency distribution of answers of pre-service teachers
to the convergence / divergence of series

To the convergence / divergence of harmonic series	*Reasons*	*n*	*Total*
Harmonic Series is Divergent	A_{21}	19	38
	A_{22}	19	
Harmonic Series is Convergent	A_{23}	9	10
	A_{24}	1	
Not Specified	A_{25}	2	2

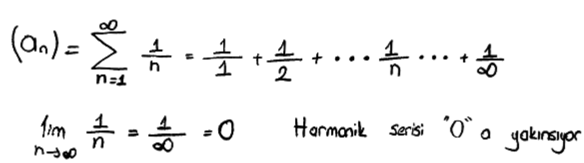
While half of the pre-service teachers asserted that harmonic series is divergent (n=19) and supported their assertions with mathematical expressions, the other half (n=19) explained their opinions with verbal expressions (Figure 3).



Translation of handwriting – "This series is an increasing series. For it to be convergent, it needs to be bounded from above. Since it is not bounded from above, it is divergent.

Figure 3. A response about divergence of harmonic series

Almost all the pre-service teachers asserted that harmonic series is convergent (n=9) and showed that the general term of the series is zero (Figure 4), while the rest (n=1) explained with verbal expressions.



Translation of the handwriting – "Harmonic series converges to 0."

Figure 4. A response about the convergence of the harmonic series

Similar situations emerged in the discussions on the convergence of the harmonic series, and the pre-service teachers verbally explained that the limit is zero as it is presented below. Following excerpt illustrates this situation:

PST_3: Since the limit of the general term of the harmonic series is zero, it is convergent.

I: So, could we say that for the general term of a series having a limit is both a necessary and a sufficient condition for convergence?

PST_3: Yes, we could …

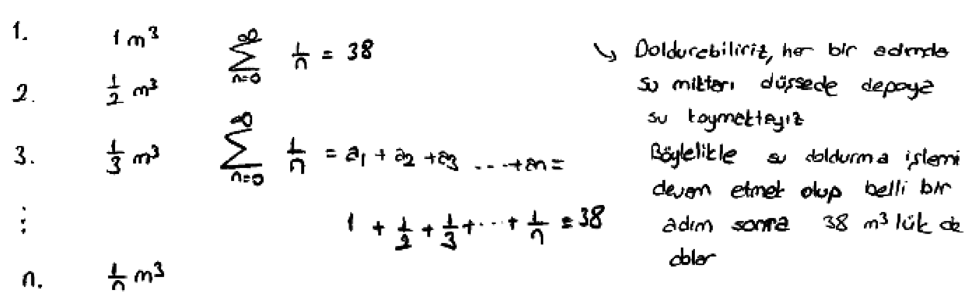
PST_4: For a series to be convergent, the limit of its general term must absolutely be 0. However, that is not sufficient on its own. The limit of harmonic series equals to 0 and yet, the harmonic series is not convergent …

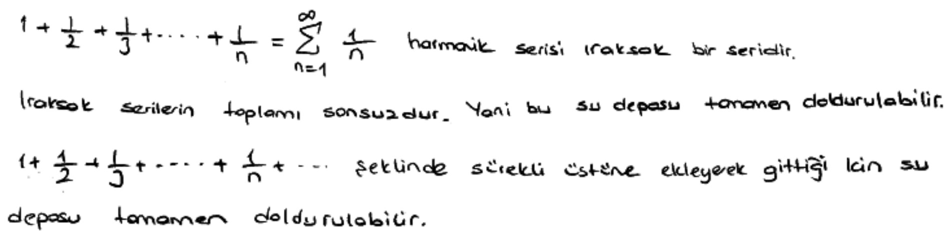
3.3. *Findings on Convergence and Divergence of Series in the Context of a Real-life Problem*

The answers given to the real-life problem will be reported in two sections. Firstly, answers of the pre-service teachers to whether the water tank can be filled or not will be presented. Then, the pre-service teachers' use of mathematical symbols / expressions / concepts in the problem-solving process will be presented. Likewise, the answers to the interview questions will be explained in the same way.

3.3.1. *Examination of the responses for fillable / unfillable water tank*

Approximately one fourth (n=13) of the pre-service teachers claimed that the water tank can be filled up (Figure 5).





Translation of handwriting – Above: *"We can fill it up. Although the water level decreases in each step, we are still adding water to the tank. And so, the action of filling continues and after a certain number of steps, the volume of 38m³ will be filled up"*. Below: *"...The harmonic series is a divergent series. The sum of divergent series equals to infinity. Thus, this water tank can be filled up completely. As the series* $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$ *constantly increases, the water tank can be filled up completely"*.

Figure 5. Responses of the pre-service teachers explaining that the water tank can be filled

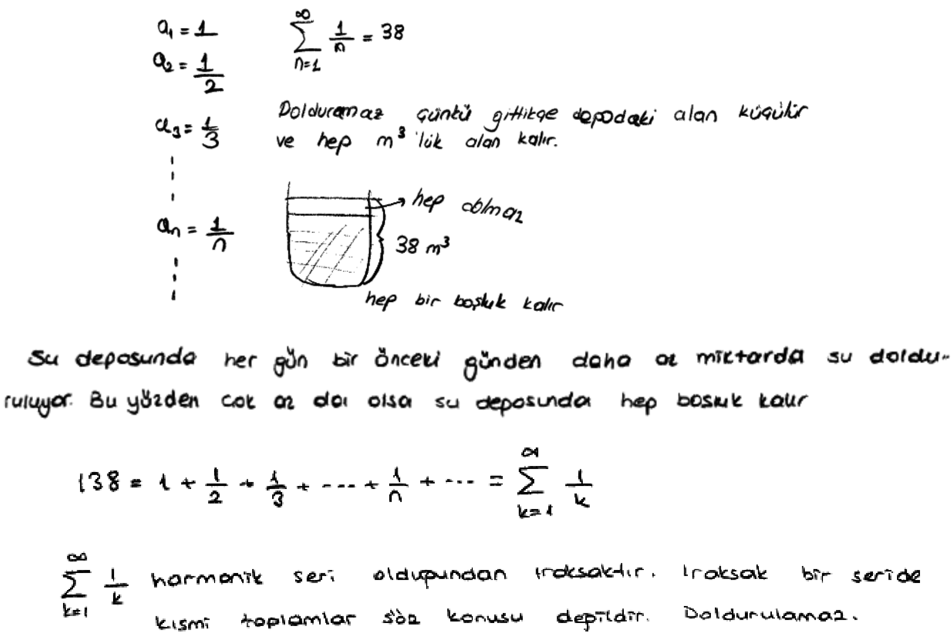
Those who claimed that the tank can be filled up explained that the sum of harmonic series reaches infinity and thus the water tank can be filled up under any condition and that as the action of filling the water tank continues infinitely, the water tank can be filled up.

TABLE V
Frequency distribution of answers of pre-service teachers to the water tank problem

Water Tank	*Reasons*	*n*	*Total*
The water tank can be filled	Harmonic series is infinite	5	13
	It continues infinitely	8	
The water tank cannot be filled	Harmonic series is divergent	12	25
	A small amount will always remain empty	10	
	There will not be a sum.	3	
Not specified	The result cannot be determined, it reaches to infinity	12	12

Half of the pre-service teachers (n=25) asserted that the water tank cannot be filled up (Figure 6). Those who asserted that the water tank cannot be filled up explained that "harmonic series is divergent, and so the tank cannot be filled", "the water tank will almost be filled but can never be entirely filled" and "there will always be a small portion left" (Table V). The remaining pre-service teachers either claimed that “there cannot be a sum”, “the series will diverge to 1 (as there will not be a limit) and so the tank will not be filled up” or they did not specify any conclusions at all.

Almost one fourth of the pre-service teachers (n=12) did not specify any conclusions regarding whether the water tank can be filled or not. They noted that “the resulting sum cannot be determined with certainty”, “as it reaches infinity, whatever happens in infinity cannot be discovered”, and that “this situation could be considered as a paradox”.



Translation of handwriting – Above: *“It cannot be filled because the volume in the tank will gradually decrease and there will always be m*^{3} *area left.”* Next to the tank figure: *“It will not constantly fill up. There will always be a space left. 38m*^{3} *in total.”* Middle: *“The amount of water added to the tank decreases each day compared to the day before. And therefore, there will always be a space left in the water tank, no matter how small.”* Below: *“[The formula] is divergent as it is a harmonic series. Partial sums are not possible in divergent series. It cannot be filled.”*

Figure 6. Responses of the pre-service teachers explaining that the water tank cannot be filled

The pre-service teachers who participated in the interviews qualified the amount of water in the water tank as infinite. Following excerpt illustrates this situation:

I: What can you say about the amount of water collected in the water tank?

PST_1: The water continues to flow, but it flows in smaller amounts. It is infinite situation.

I: Can you please further explain what you think about the infinite situation?

PST_1: The action of filling water will continue infinitely, it still continues even if in smaller amounts.

I: So, can the tank be filled up?

PST_1: It will continue infinitely, it must fill it up eventually.

I: And what if the water flowing to the water tank was expressed as $\frac{1}{2}m^3$, $\frac{1}{4}m^3$, $\frac{1}{8}m^3$, ... , $\frac{1}{2^n}m^3$, ... ? Could the tank then be filled up?

PST_1: Yes, it could still be filled up, it still reaches infinity...

I: Do you think there may be other criteria beside to this for the water tank to be filled up?

PST_1: No, not actually. It can be filled if it is filled infinitely.

I: What can you say about the amount of water collected in the water tank?

PST_3: When I examine the amount of water flow, I conclude that there is a constant, yet slow flow of water.

I: So, can the water tank be filled up?

PST_3: No, it cannot be filled up because there will always be a space left.

I: Why will there be a space left?

PST_3: Even though we will continuously fill the tank, the amount of added water will decrease constantly, and so, there will always be a space left. Also, it will never be entirely empty…

When the amount of water flowing into the water tank was expressed as a geometric series, only PS_4 stated that whether the water tank can be filled or not will change as the speed of water flowing to the water tank changes, while other pre-service teachers maintained their previous opinions. Following excerpt illustrates this situation:

I: If the action of filling was expressed as $\frac{1}{2}m^3$, $\frac{1}{4}m^3$, $\frac{1}{8}m^3$, ... , $\frac{1}{2^n}m^3$, ... , could the water tank then be filled?

PST_3: No, the same logic still applies. There will still be a small space left. There will always be an empty space left in the water tank regardless of the amount of water like this…

PST$_2$: This series is not a harmonic series. I need to think this over. Hmm… (makes some calculations) This is a convergent series. So, if we calculate the result…It cannot be filled because the result equals to 1. So, there will only be 1m^3 water collected in the water tank.

I: So, why the water tank cannot be filled when the series is harmonic?

PST$_2$: I mean, when I think it over now, I think it should be filled then. Because it is a harmonic series.

I: Could you please further explain your opinion?

PST$_2$: Well, I think I miscalculated on paper. Because the water tank can indeed be filled when it is a harmonic series as it is greater than a real number. When it is a geometric series, the sum equals to 1.

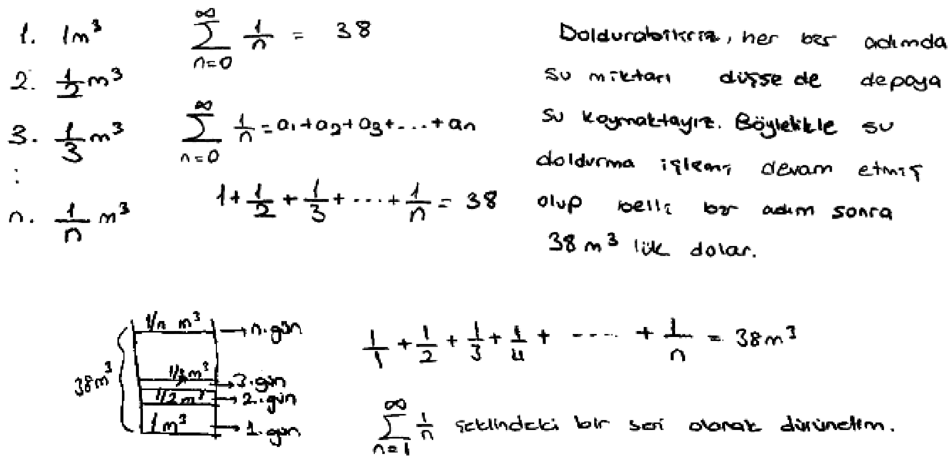
3.3.2. *Examination of responses in terms of mathematical symbol / expression / concepts.*

Almost half of the pre-service teachers (n=24) noted that the expression $1\frac{m^{3,1}}{2}\,\frac{m^{3,1}}{3}$ m^3, ..., $\frac{1}{n}$ m^3, ... provided in P_1 is a harmonic series. More than two thirds of the pre-service teachers who affirmed it to be a harmonic series (n=17) claimed that it is a divergent series, while a minority (n=3) claimed that the harmonic series is convergent (Table VI).

TABLE VI
Frequency distribution of answers of the pre-service teachers to the water tank problem on the basis of harmonic series

The sum of the expression	*Reasons*	*n*	*Total*
It is a Harmonic Series	It is Divergent	17	24
	It is Convergent	3	
	Not Specified	4	
Not Specified	It is Divergent	6	26
	Not Specified	20	

More than half of the pre-service teachers (n=26) claimed that the sum of the expression in P_1 is infinite, and while they expressed various concepts of series and sequences, they did not specify any conclusions regarding whether it is a harmonic series. Almost one fourth of pre-service teachers (n=6), on the other hand, claimed that it is a divergent series.



Translation of handwriting – Above: *"We can fill it up; while the water level decreases in each step, we are still adding water to the tank. And so, the action of filling continues and after a certain number of steps, the volume of 38m³ will be filled up."* Below: *"[in the figure] 1st day, 2nd day, 3rd day, nth day"; "let's consider it as a series expressed in [formula]".)*

Figure 7. Responses of the pre-service teachers by using representation in the water tank problem

All the pre-service teachers used ten different representations referring to the series and sequences (Table VII) once or more than once while solving P_1 (Figure 7). Therefore, the number of representations referring to the concepts was calculated as "n=69" in Table VII. As can be observed in Table VII, in the solution of water tank problem, the pre-service teachers used various representations such as series, series equalling to a number or infinity, write or not writing subscripts and superscripts of series, finite expression of series, sequences, and sequence of partial sums.

TABLE VII

Representations used by the pre-service teachers in the solution of water tank problem

Code	*Expression / Representation*	*n*	*Explanation*
C_1	$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + ... + \frac{1}{n} + ... = \infty$	5	The sum of series equals to infinity
C_2	$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + ... + \frac{1}{n} + ... = {}_?138\ m^3$	4	Explicit formula of series and using the sum of the series in the problem

C_3	$\sum_{n=1}^{\infty}\frac{1}{n} = {?}138\ m^3$	6	Using of the sum of the series in the problem
C_4	$\sum_{n=1}^{\infty}\frac{1}{n}$	9	Expression of series with its indices
C_5	$\sum\frac{1}{n}$	7	Expression of series without its indices
C_6	$1+\frac{1}{2}+\ldots+\frac{1}{n} = 138\ m^3$	5	Consideration of the terms of series as finite and using in the problem
C_7	$1+\frac{1}{2}+\ldots+\frac{1}{n}$	7	Expression of series as finite
C_8	$a_1 = 1\ \ a_2 = \frac{1}{2}\ \ldots\ a_n = \frac{1}{n}$	17	Definition of the first n term of the series
C_9	$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\ldots+\frac{1}{n}$	7	Expression of the series with its indices, and only as the sum of the n term
C_{10}	$\sum a_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}$	2	Expression of the series without its indices, and only as the sum of the n term

The series was represented as an infinite sum expressed in C_1 or C_2 only by a minority of pre-service teachers. On the other hand, the pre-service teachers generally expressed that when the water tank is filled, either an infinite calculation takes place as C_8 or that the sum of series as C_9 is finite.

One of the pre-service teachers confused the concepts of series and sequences when she was asked to explain the solution of the problem in the interview. The pre-service teacher used property of sequences while expressing series. That is, she stated that $\frac{1}{n}$ is the last term of a series but actually it is the last term of a sequence. Following excerpt illustrates this situation:

I: Can you repeat the solution of the given question?
PST_4: [Starts to do the calculations]
I: Can you please further explain this representation? (indices are shown)
PST_4: It means this. So, from one to infinity.
I: Then, why did not you specify this in your solution?
PST_4: I do not really care that. I think it makes little difference whether I write it.
I: So, can you explain what you mean here? (the first n term of the series is shown)
PST_4: I expanded the series. It continues as the first term, second term, third, fourth and the n^{th} term.

I: I see. Then, what is the nth term in this expansion?
PST_4: As the nth term is the last term, then it is $a_n=\frac{1}{n}$.
I: So, the last term is $a_n=\frac{1}{n}$?
PST_4: Yes.

Others were more careful in their use of the indices of series. Two of the pre-service teachers considered the series to be finite.

4. Discussion, conclusion and suggestions

In this study, the knowledge of pre-service elementary mathematics teachers on convergence and divergence of series and how they apply this knowledge to the problem enriched with real-life context were examined. Findings have revealed that pre-service teachers performed better in explaining theoretical aspect of the convergence of series than in explaining the divergence of series. One of the reasons for this situation may be the fact that the concepts such as theorems, results and criteria that the pre-service teachers have learned are mostly built on the concept of "convergence" in textbooks (Thomas, Weir, Hass & Heil, 2014). The fact that the convergence and divergence concepts are used as opposite may arise from the fact that these two are dichotomous concepts (Sfard, 1991).

The fact that the pre-service teachers' explanations on the convergence of series focus on the sequence of partial sums and on the Cauchy sequence implies that they associate the concept of sequences with series. In fact, the pre-service teachers explained the convergence of sequence instead of the convergence of series (Mamona, 1990); thus, it can be said that series and sequences are seen by the pre-service teachers as concepts that are considered together due to their nature. In this study, the pre-service teachers misused the concepts of series and sequences in each other's place and these findings were consistent with the findings of the studies conducted by Alcock and Simpson (2004). The misuse of the series and sequence concepts could be related to mislearning or concept confusions as well as pre-service teachers' lack of knowledge about limit, infinity concepts and their carelessness in the use of representations and mathematical language. Pre-service teachers had a difficulty when they were exposed to a challenging real-life problem because the problem involved both limit concept and infinity of series. Therefore, they gave intuitive responses based on their imaginations of infinite series (Barahmand, 2020), limit and sequence. Another indicator of pre-service teachers' confusion of series and sequence is that they expressed the amount of water flow into the water tank by using sequence representations or that they expressed series as finite while they are solving the problem because of the limited understanding of limit.

Most pre-service teachers accurately determined the character of harmonic series and they correctly supported their conclusions with mathematical expressions and verbal expressions. Those who claimed that harmonic series is convergent reasoned that the limit of the general term of $a_n=\frac{1}{n}$ equals to zero. This could originate from the fact that "harmonic series are among the most popular series whose general term converges to zero, but the series itself diverges" (Aliyev & Dil, 2012). Even the statement in textbooks and courses that harmonic series is divergent despite having a limit did not change the belief of pre-service teachers. It could be suggested that this belief stems from the perception that the opposite of the "*If a series is convergent, then its general term equals to zero*" proposition must also be true. In addition to their perception of divergence as an opposite of convergence, the pre-service teachers supported their claims that harmonic series is divergent with mathematical and verbal expressions which could be interpreted as accurate theoretical knowledge. However, the pre-service teachers interpreted differently their theoretical knowledge in the water tank problem. The fact that the divergence of harmonic series was interpreted in two distinct aspects in the water tank problem may indicate that the concept is not fully comprehended by the pre-service teachers.

The pre-service teachers who claimed that the water tank can be filled because the harmonic series is divergent adopted a correct approach in application of their theoretical knowledge on the divergence of harmonic series to practice. This approach may stem from the belief that the sum of harmonic series equals to infinity. This may suggest that similar to the studies of Bagni (2000) and Fiscbein, (1987), the pre-service teachers could associate the concept of infinity with the amount of water flow. On the other hand, the pre-service teachers who claimed that the water tank cannot be filled because harmonic series is divergent did not adopt a correct approach in application of their theoretical knowledge on the divergence of harmonic series to practice. This approach may stem from divergence of harmonic series and the sum of divergent series not being equal to a real number and instead, being equal to infinity.

Various inferences of the pre-service teachers regarding the water tank problem and the various bases for their explanations and their solutions are related to the differences in their application of their theoretical knowledge to the practice. Even though some of the pre-service teachers modelled the solution of the problem by using some drawings, notations or symbols related to the amount of water flow with harmonic series, they claimed that the water tank cannot be filled completely because that harmonic series is divergent underlines the necessity to focus primarily on the concept of "divergence". Thus, it can be said that they could not make a transition from theoretical knowledge to practice.

The fact that the pre-service teachers do not consider the amount of water flowing into the water tank as "infinite" may cause challenges in interpreting the flowing water (Fischbein, 1987). Another significant problem the pre-service teachers faced in the comprehension of the infinity concept stems from the perception that "infinite terms imply infinite sum" (Bagni, 2000; 2005; 2008). While the amount of water flowing into the water tank was modelled as a geometric series in the interviews, the pre-service teachers claimed that "the water tank will be filled as the amount of water flowing from the tap continues infinitely". This leads pre-service teachers to have difficulties in building the idea of infinity of series. Pre-service teachers who perceived the amount of water flowing into the water tank as infinite and claimed that the tank can be filled accurately solved the real-life problem with this approach. Specifying the amount of water as infinite and that the tank can be filled in the problem solving process can be considered as a correct approach in the application of theoretical knowledge to practice.

Incorrect use of expressions / representations, lack of indices, or not using equations in the problem-solving process may be related to the pre-service teachers' prior knowledge while reflecting their opinions on the paper and not pondering properly on their writing processes. The misuse of various mathematical expressions / representations, confusion of series and sequence concepts as well as the expression of infinity with the infinity symbol in the solution of the water tank problem were the other findings that emerged in the present study. Besides, while some of the pre-service teachers were able to notice and correct their errors during the interviews, others did not correctly use representations / expressions during the interviews and they were careless in the use of mathematical language. Therefore, importance should be given to the correct use of mathematical language (Raiker, 2002) in the process of teaching series. Furthermore, the attention of the pre-service teachers on their use of indices while they are repeating the solution of the problem in interviews suggests that the pre-service teachers are careless while they are writing series and sequences.

In conclusion, this study aimed to examine the knowledge of pre-service teachers on the convergence and the divergence of series within the context of theory and practice. The concept of series is considered as a complex structure (González-Martín, Nardi & Biza, 2011) and is seemed as difficult (Cornu, 1991; González-Martín, 2013b; Lindaman & Gay, 2012). The fact that the harmonic series is a divergent series despite its limit being zero was a challenging point for pre-service teachers. Therefore, in the teaching process of harmonic series it can be suggested to highlight the relationship between limit and divergence of harmonic series. The limited expression of pre-service teachers about series, determining only the first n term of the series while expressing the series, and explaining the convergence of sequence instead of convergence of series reveal that they confuse the concepts of series and sequences (Schwarzenberger & Tall, 1978). Thus, it might be suggested to underline the relationship between series and other concepts of Calculus such as sequences in the teaching of this concept.

In the application of theoretical knowledge to practice, the water tank problem raised issues such as the convergence and divergence of harmonic series, whether the water tank can be filled or not and the different interpretations of the amount of collected water in the tank. It could be suggested that problems become clearer when they are considered within the context of their solutions (Ergene, 2014). Therefore, in identifying the amount of water flow by the pre-service teachers, various meanings of series, harmonic series, divergence and infinity (Bagni 2008; 2000; Fischbein,1987) were influential on the different interpretations of the water tank problem. This might stem from difficulty in making abstract concepts of mathematics such as divergence and infinity concrete and in associating them with real-life problems in the teaching process (Ergene, 2019). Therefore, using real-life problems such as the turtle paradox (Duran, Doruk & Kaplan, 2016) and the water tank problem that was used in this study in the teaching process of series may help pre-service teachers apply theoretical knowledge on divergence and infinity and other abstract concepts to practical knowledge. This case also underlines the need to focus on limit and infinity while teaching the divergence of series in order to eliminate the difficulties in the interpretation of the limit and infinity concepts (Fischbein, 2001; Tall, 1980; Tsamir & Tirosh, 1999). Thus, it is recommended that activities related to sequences and series enriched with real-life problems based on the concepts of divergence and infinity can be integrated to Calculus courses. In addition, similar studies can be conducted with pre-service secondary and in-service mathematics teachers in order to compare and contrast the findings. Finally, conducting research studies which focus on the harmonic series might also be suggested when the studies related to Taylor Series (Tokgöz, 2019; Martin, 2012) and Power Series (Kung & Speer, 2010) were taken into account.

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Authors

Özkan Ergene. Sakarya University, Faculty of Education, Turkey, ozkanergene@sakarya.edu.tr

Ahmet Şükrü Özdemir. Marmara University, Atatürk Faculty of Education, Turkey, ahmet.ozdemir@marmara.edu.tr

Appendix I

Q_1, Q_2, P_1 which are used in the research process and their answers.

Code	*Question and Answer*
Q_1	What does it mean for a series to be convergent or divergent? Please explain.
A_{11}	$(a_n)=\sum_{n=1}^{\infty} a_n$ sequence of partial total. $S_1=a_1$, $S_2=S_1+a_2=a_1+a_2$ $S_3=S_2+a_3=a_1+a_2+a_3$ … $S_n=S_{n-1}+a_n=a_1+a_2+a_3+\cdots+a_n$ … The $(a_n)=\sum_{n=1}^{\infty} a_n$ series is convergent too if the $(S_n)=(S_1, S_2, ..., S_n, ...)$ sequence of partial sums converged by the S_n partial sum
A_{12}	The necessary and sufficient condition for the convergence of the $\sum_{n=1}^{\infty} a_n$ series is that there is a $n(\varepsilon)\in N$ natural number as $\left\|\sum_{k=m+1}^{n} a_k\right\|<\varepsilon$ when $n(\varepsilon)<m<n$ corresponding to each number of $\varepsilon\in R^+$
A_{13}	The necessary and sufficient condition for the $\sum_{n=1}^{\infty} a_n$ series to be convergent is that the (S_n) sequence of sums is a Cauchy sequence.
A_{14}	Criteria of Comparison - D'alembert Ratio Test - Cauchy Root Test - Raabe Test - Alterne Convergency Test
A_{15}	Verbal expressions
A_{16}	Other Answers
Q_2	Examine the convergence of the $(a_n)=\sum_{n=1}^{\infty}\frac{1}{n}$ harmonic series. (Aliyev & Dil, 2012)

A_{21}	Harmonic series is divergent, representation with mathematical expressions.
A_{22}	Harmonic series is divergent, with verbal expressions.
A_{23}	Harmonic series is convergent, with mathematical expressions.
A_{24}	Harmonic series is convergent, with verbal expressions.
A_{25}	Other Answers
P_1	A water tank with 38 m^3 capacity is required to be filled as follows; - First day $1m^3$, - Second day $\frac{1}{2}m^3$, - Third day $\frac{1}{3}m^3$, ... , - nth day $\frac{1}{n}m^3$, ... Can water tank completely can be filled in this way? Explain your answer with the reasons.
P_{11}	Water tank can be filled / cannot be filled.
P_{12}	Specifying/not specifying that it is a harmonic series and convergence / divergence of the harmonic series.
P_{13}	Examination of answers in the context of mathematical symbol / expression / concepts.