T.R.
SAKARYA UNIVERSITY
INSTITUTE OF SCIENCE AND TECHNOLOGY

SOME IDENTITIES AND DIOPHANTINE EQUATIONS INCLUDING GENERALIZED FIBONACCI AND LUCAS NUMBERS

Ph.D. THESIS

Zafer ŞİAR

Department	**:**	**MATHEMATICS**
Field of Science	**:**	**ALGEBRA AND NUMBER THEORY**
Supervisor	**:**	**Prof. Dr. Refik KESKİN**

December 2012

T.R.
SAKARYA UNIVERSITY
INSTITUTE OF SCIENCE AND TECHNOLOGY

SOME IDENTITIES AND DIOPHANTINE EQUATIONS INCLUDING GENERALIZED FIBONACCI AND LUCAS NUMBERS

Ph.D. THESIS

Zafer ŞİAR

Department : MATHEMATICS

This thesis was adopted unanimously by the following jury on 14 / 12 /2012.

Prof. Dr. Refik KESKİN	**Prof. Dr. Halim ÖZDEMİR**	**Assoc. Prof. Dr. Mehmet BEKTAŞOĞLU**
Jury Chairman	**Member**	**Member**
Assoc. Prof. Dr. Neşe ÖMÜR		**Assoc. Prof. Dr. Yücel TÜRKER ULUTAŞ**
Member		**Member**

ACKNOWLEDGEMENTS

I would like sincerely to thank my supervisor Refik KESKİN, who guided me with great patience during the process of writing my thesis. Moreover, I would like to thank the Scientific and Technical Research Council of Turkey (TÜBİTAK) for financial support during my doctorate programme.

This thesis is supported by Commission for Scientific Research Projects of Sakarya University (Project Number 2011-50-02-019).

TABLE OF CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

N	: The set of natural numbers
Z	: The set of integers
R	: The set of real numbers
$a \mid b$	: a divides b
$a \nmid b$	: a does not divide b
(a,b)	: The greatest common divisor of a and b
$[a,b]$	: The least common multiple of a and b
$\left(\frac{\cdot}{\cdot}\right)$	: Jacobi Symbol
$\{F_n\}$	: Fibonacci sequence
$\{L_n\}$	: Lucas sequence
$\{P_n\}$	: Pell sequence
$\{Q_n\}$	: Pell-Lucas sequence
$\{U_n\}$	: Generalized Fibonacci sequence
$\{V_n\}$	: Generalized Lucas sequence
$\sum$	: Summation symbol
$a \parallel b$	: $a \mid b$ and $(a,b/a)=1$

SOME IDENTITIES AND DIOPHANTINE EQUATIONS INCLUDING GENERALIZED FIBONACCI AND LUCAS NUMBERS

SUMMARY

Key Words: Fibonacci and Lucas Numbers, Generalized Fibonacci and Lucas Numbers, Congruences, Diophantine Equations.

In the first chapter, firstly, Fibonacci and Lucas numbers are mentioned briefly. Also the definitions of the generalized Fibonacci and Lucas sequences are given. Then, the review of the literature concerning generalized Fibonacci and Lucas sequences are given.

In the second chapter, some identities and summation formulas containing generalized Fibonacci and Lucas numbers are obtained. Some of them are well known while the remaining ones new. Using some of these identities and summation formulas, it is given some congruences concerning generalized Fibonacci and Lucas numbers such as

$$V_{2mn+r} \equiv (-(-Q)^m)^n V_r \left(mod V_m\right), U_{2mn+r} \equiv (-(-Q)^m)^n U_r (mod V_m),$$

and

$$V_{2mn+r} \equiv (-Q)^{mn} V_r \left(mod U_m\right), U_{2mn+r} \equiv (-Q)^{mn} U_r \left(mod U_m\right).$$

Fibonacci and Lucas numbers of the form cx^2 are determined after some fundamental theorems and identities concerning Fibonacci and Lucas numbers are given in the third chapter.

In the fourth chapter, generalized Fibonacci and Lucas numbers of the form cx^2 are determined under some assumptions using congruences concerning generalized Fibonacci and Lucas numbers given in the second chapter.

GENELLEŞTİRİLMİŞ FİBONACCİ VE LUCAS SAYILARINI İÇEREN BAZI ÖZDEŞLİKLER VE DİOFANT DENKLEMLERİ

ÖZET

Anahtar Kelimeler: Fibonacci ve Lucas Sayıları, Genelleştirilmiş Fibonacci ve Lucas Sayıları, Kongrüanslar, Diofant Denklemleri.

İlk bölümde, ilk olarak, Fibonacci ve Lucas sayılarından kısaca bahsedilmiştir. Ayrıca, genelleştirilmiş Fibonacci ve Lucas dizilerinin tanımları verilmiştir. Sonra genelleştirilmiş Fibonacci ve Lucas dizileriyle ilgili literatür özeti verilmiştir.

İkinci bölümde, genelleştirilmiş Fibonacci ve Lucas sayılarını içeren bazı özdeşlikler ve toplam formülleri elde edilmiştir. Bunların bazıları yenidir ve bazıları da iyi bilinir. Bu özdeşliklerin ve toplam formüllerinin bazıları kullanılarak,

$$V_{2mn+r} \equiv (-\left(-Q\right)^{m})^{n} V_{r} (mod V_{m}),\ U_{2mn+r} \equiv (-\left(-Q\right)^{m})^{n} U_{r} (mod V_{m})$$

ve

$$V_{2mn+r} \equiv \left(-Q\right)^{mn} V_{r} \left(mod U_{m}\right),\ U_{2mn+r} \equiv \left(-Q\right)^{mn} U_{r} \left(mod U_{m}\right)$$

gibi genelleştirilmiş Fibonacci ve Lucas sayılarını içeren bazı kongrüanslar verilmiştir.

Üçüncü bölümde, Fibonacci ve Lucas sayılarını içeren bazı temel teoremler ve özdeşlikler verildikten sonra cx^2 formunda olan Fibonacci ve Lucas sayıları tespit edilmiştir.

Dördüncü bölümde ise bazı şartlar altında cx^2 formunda olan genelleştirilmiş Fibonacci ve Lucas sayıları, ikinci bölümdeki genelleştirilmiş Fibonacci ve Lucas sayılarını içeren kongrüanslar kullanılarak tespit edilmiştir.

CHAPTER 1. INTRODUCTION

The Italian mathematician Leonardo Fibonacci is considered as “the most talented western mathematician of the Middle Ages”. Fibonacci’s mathematical background began during his many visits to North Africa, where he was introduced to early works of algebra, arihtmetic and geometry. He also travelled to countries located in the Mediterranean region and studied the mathematical systems that were practicing. His travels led him to the realization that Europe was lacking on the mathematical scene.

After widespread travel and extensive study of computational systems, Fibonacci wrote the *Liber Abaci* in 1202, in which he explained the Hindu-Arabic numerals and how they were used in computation.

Although he wrote on a variety of mathematical topics, Fibonacci is remembered particularly for the sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

which is known today as Fibonacci sequence. The elements of Fibonacci sequence are called Fibonacci numbers and nth Fibonacci number is represented by F_n. These numbers satisfy the relation

$$F_{n+1} = F_n + F_{n-1}$$

for $n \geq 1$ with $F_0 = 0$, $F_1 = 1$. Fibonacci sequence is related to closely many number sequences such as Lucas sequence. Lucas sequence,

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots,$$

was introduced by François Edouard Anatole Lucas, a French mathematician. The elements of Lucas sequence are called Lucas numbers. nth Lucas number is represented by L_n and these numbers satisfy the relation

$$L_{n+1} = L_n + L_{n-1}$$

for $n \geq 1$ with $L_0 = 2,\ L_1 = 1$. In fact, this two sequences are related to each other by hundreds of identities.

Many scientist, especially mathematicians, deal with Fibonacci and Lucas sequences. Because Fibonacci and Lucas numbers are seen in many areas such as in nature, some of the historic buildings, some music instruments, and physics. For example, in nature, pinecones and sunflowers display Fibonacci numbers in a unique and remarkable way. The seeds of sunflowers occur in spirals, one set of spirals going clockwise and one set going counterclockwise. The most common number of this spirals are 34 in one direction and 55 in the other. Consecutive Fibonacci numbers also appear as the number of spirals formed by the scales of pinecones. Moreover, the number of petals in many flowers such as iris, trillium, bluet, wild rose, hepetica, blood root, and cosmos, is often a Fibonacci number. In music, an octave is an interval between two pitches, each of which is represented by the same musical note. On the piano's keyboard, an octave consist of 5 black keys and 8 white keys, totaling 13 keys. In addition, the black keys are divided into a group of two and a group of three keys. Besides, there are a close relationship between Fibonacci (or Lucas) sequence and golden ratio. It is well known that as n gets larger and larger, the ratio F_{n+1}/F_n (or L_{n+1}/L_n) approaches the golden ratio $\left(1+\sqrt{5}\right)/2$.

On the other hand, Fibonacci and Lucas numbers have many interesting properties. In many studies, it is given the summation formulas, divisibility properties, congruences and also many identities concerning sequences of these numbers. Some congruences concerning Fibonacci and Lucas numbers are given in the following:

$$F_{2mn+r} \equiv (-1)^{mn} F_r \left(mod F_m\right),$$

$$L_{2mn+r} \equiv (-1)^{mn} L_r \left(mod F_m\right),$$

$$L_{2mn+r} \equiv (-1)^{(m+1)n} L_r \left(mod L_m\right),$$

and

$$F_{2mn+r} \equiv (-1)^{(m+1)n} F_r \left(mod L_m\right)$$

for all $n, m \in \mathbf{N} \cup \{0\}$ and $r \in \mathbf{Z}$ [22]. Moreover, some studies on the divisibility properties of F_n and L_n have been made. For example, it was shown that if $m \mid n$, then $F_m \mid F_n$. Then, in 1964, L. Carlitz established the converse of this case, that is, if $F_m \mid F_n$, then $m \mid n$. Moreover, in [5], L. Carlitz showed the following two divisibility properties:

a) $L_m \mid F_n$ if and only if $2m \mid n$, where $m \geq 2$.

b) $L_m \mid L_n$ if and only if $n = (2k+1)m$, where $m \geq 2$ and $k \geq 0$.

These divisibility properties were also investigated in [15], [16], and [48]. Also, the proofs of these divisibility properties were done in [22] using the congruences given above.

Besides, while some summation formulas containing Fibonacci and Lucas numbers were found, the Fibonacci matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

was studied by Charles H. King in 1960 for his master thesis [25], and some other matrices were used. Using these matrices, many identities concerning Fibonacci and Lucas numbers are obtained. In fact, if $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then it can be seen that $\mathbf{A}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$. Thus, from the matrix equality $\mathbf{A}^{m+n} = \mathbf{A}^m \mathbf{A}^n$, it is obtained the identities

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n,$$

$$F_{m+n} = F_{m+1}F_n + F_m F_{n-1},$$

and

$$F_{m+n-1} = F_m F_n + F_{m-1}F_{n-1}.$$

Then, using these identities, it is obtained the identities

$$F_{m+1}L_n + F_m L_{n-1} = L_{m+n},$$

$$F_m L_n + F_n L_m = 2F_{m+n},$$

and

$$L_m L_n + 5F_m F_n = 2L_{m+n} .$$

Other than the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ in [21], the authors used the matrix $\mathbf{S} = \begin{bmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{bmatrix}$ and they showed that $\mathbf{S}^n = \begin{bmatrix} L_n/2 & 5F_n/2 \\ F_n/2 & L_n/2 \end{bmatrix}$. Using this property and the fact that $\mathbf{S}^2 = \mathbf{S} + \mathbf{I}$, the authors obtained some identities concerning Fibonacci and Lucas numbers.

Moreover, many mathematicians are interested in determining the Fibonacci and Lucas numbers which are a perfect square or twice a perfect square. Using the divisibility properties of F_n and L_n and congruences given above, Fibonacci and Lucas numbers which are a perfect square or twice a perfect square are determined. Historically, we will summarize studies in this subject in the next. Besides, determining Fibonacci and Lucas numbers of the form x^2 and $2x^2$ is facilitated in the solution of many Diophantine equations. For example, it is well known that all positive integer solutions of the equations

$$x^2 - 5y^2 = \mp 4$$

and

$$x^2 - xy - y^2 = \mp 1$$

are given by $(x, y) = (L_n, F_n)$ and $(x, y) = (F_{n+1}, F_n)$ with $n \geq 1$, respectively. Thus, it can be easily found all positive integer solutions of the equations

$$x^4 - 5y^2 = \mp 4,\ x^2 - 5y^4 = \mp 4,\ 4x^4 - 5y^2 = \mp 4,$$

$$x^4 - x^2 y - y^2 = \mp 1,\ x^2 - xy^2 - y^4 = \mp 1,$$

since Fibonacci and Lucas numbers of the form x^2 and $2x^2$ are known. For more information about Fibonacci and Lucas numbers, one can consult [26] and [48].

The studies mentioned above have been made for generalized Fibonacci and Lucas sequences, too.

In [17], Horadam defined a sequence as follows:

$$W_0 = a,\ W_1 = b \text{ and } W_{n+1} = W_{n+1}(a, b, P, Q) = PW_n - QW_{n-1}$$

for $n \geq 1$, where $a, b, P, Q \in \mathbf{Z}$. Particular cases of the sequence (W_n) are sequences (F_n), (L_n), (U_n), and (V_n) given by

$$W_n(0,1,P,-Q) = U_n(P,Q),$$

$$W_n(2,P,P,-Q) = V_n(P,Q),$$

$$W_n(0,1,1,-1) = F_n,$$

and

$$W_n(2,1,1,-1) = L_n,$$

respectively. Thus, the sequence (U_n) called generalized Fibonacci sequence satisfies the recurrence relation $U_{n+1} = U_{n+1}(P,Q) = PU_n + QU_{n-1}$ for $n \geq 1$ with $U_0 = 0$, $U_1 = 1$ and the sequence (V_n) called generalized Lucas sequence satisfies the recurrence relation $V_{n+1} = V_{n+1}(P,Q) = PV_n + QV_{n-1}$ for $n \geq 1$ with $V_0 = 2$, $V_1 = P$. Of course, the sequences (U_n) and (V_n) are generalizations of Fibonacci and Lucas sequences and the sequence (W_n) is also a different generalization of Fibonacci and Lucas sequences. But Horadam is not the first author, who defined generalized Fibonacci and Lucas sequences. The sequences (U_n) and (V_n), firstly, were introduced by Lucas in [28]. For more information about generalized Fibonacci and Lucas sequences, one can consult [20], [28], [33], [37], and [41].

U_n and V_n are called nth generalized Fibonacci number and nth generalized Lucas number, respectively. Generalized Fibonacci and Lucas numbers for negative subscripts are given by

$$U_{-n} = \frac{-U_n}{(-Q)^n} \text{ and } V_{-n} = \frac{V_n}{(-Q)^n}, \tag{1.1}$$

respectively.

Now assume that $P^2 + 4Q > 0$. Then it is well known that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n, \tag{1.2}$$

where $\alpha = \left(P + \sqrt{P^2 + 4Q}\right)/2$ and $\beta = \left(P - \sqrt{P^2 + 4Q}\right)/2$ are the roots of the characteristic equation $x^2 - Px - Q = 0$. Clearly $\alpha + \beta = P$, $\alpha - \beta = \sqrt{P^2 + 4Q}$, and $\alpha\beta = -Q$. The formulas in (1.2) are known as Binet's formulas. Moreover, it is well known the relations

$$V_n = U_{n+1} + QU_{n-1} = PU_n + 2QU_{n-1} \tag{1.3}$$

and

$$(P^2 + 4Q)U_n = V_{n+1} + QV_{n-1} \tag{1.4}$$

for every $n \in \mathbf{Z}$ between the sequences U_n and V_n and these relations can be easily proved using Binet's formulas.

Besides, generalized Fibonacci and Lucas numbers have the following divisibility properties:

c) If $U_m \neq 1$, then $U_m \mid U_n$ if and only if $m \mid n$.

d) If $V_m \neq 1$, then $V_m \mid U_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is even.

e) If $V_m \neq 1$, then $V_m \mid V_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is odd.

These divisibility properties have been expressed in [15], [39], [40], [41], and [42].

On the other hand, generalized Fibonacci and Lucas numbers are the solutions of some Diophantine equations. For example, all positive integer solutions of the equations $x^2 - (P^2 + 4)y^2 = 4$ and $x^2 - (P^2 + 4)y^2 = -4$ are given by $(x, y) = (V_{2n}(P,1), U_{2n}(P,1))$ with $n \geq 1$ and $(x, y) = (V_{2n-1}(P,1), U_{2n-1}(P,1))$ with $n \geq 1$, respectively. And all positive integer solutions of the equation $x^2 - (P^2 - 4)y^2 = 4$ are given by $(x, y) = (V_n(P,-1), U_n(P,-1))$ with $n \geq 1$. Also all positive integer solutions of the equations $x^2 - Pxy - y^2 = 1$ and $x^2 - Pxy - y^2 = -1$ are given by $(x, y) = (U_{2n+1}(P,1), U_{2n}(P,1))$ with $n \geq 1$ and $(x, y) = (U_{2n}(P,1), U_{2n-1}(P,1))$ with $n \geq 1$, respectively. Moreover, all positive integer solutions of the equation $x^2 - Pxy + y^2 = 1$ are given by $(x, y) = (U_{n+1}(P,-1), U_n(P,-1))$ with $n \geq 1$. The solutions of these equations were given in [18], [24], [30], and [51]. Besides, all positive integer solutions of the equations

$$x^4-(P^2+4)y^2=\mp 4,\ x^2-(P^2+4)y^4=\mp 4,$$

$$x^4-(P^2-4)y^2=4,\ x^2-(P^2-4)y^4=4,$$

$$x^4-Px^2y-y^2=\mp 1,\ x^2-Pxy^2-y^4=\mp 1,$$

and

$$x^2-Pxy^2+y^4=1$$

are easily found using generalized Fibonacci and Lucas numbers, which are perfect square. Solutions of the above equations were investigated in [9], [10], and [12].

Now, we give a summary of the literature concerning generalized Fibonacci and Lucas numbers of the form cx^2.

As it is mentioned above, many mathematicians are interested in determining the Fibonacci and Lucas numbers, which are perfect square. The problem of characterizing the square Fibonacci numbers was first introduced in the book by Ogilvy [36]. In 1963, both, Moser and Carlitz [32], and Rollet [46] proposed this problem. In 1964, the square conjecture was proved by Cohn [6] and independently by Wyler [50]. Later the problem of characterizing the square Lucas numbers was solved by Cohn [8] and by Alfred [1]. Moreover, determining the Fibonacci and Lucas numbers, which are twice a perfect square, has been the subject of curiosity, too. In 1965, Cohn solved the Diophantine equations $F_n=2x^2$ and $L_n=2x^2$ in [8]. Congruences were widely used in the solution of these problems.

Besides, there has been much interest in when the terms of generalized Fibonacci and Lucas sequences are perfect square or k times a square. Now we summarize here results on this problem. Firstly, in [27], Ljunggren showed that for $n\geq 2$, P_n is a perfect square precisely for $P_7=13^2$, and $P_n=2x^2$ precisely for $P_2=2$. In [9, 10], Cohn solved the Diophantine equations $U_n=x^2$, $2x^2$ and $V_n=x^2$, $2x^2$ with odd P and $Q=\pm 1$. Moreover, in [39], Ribenboim and McDaniel determined all indices such that for all odd relatively prime integers P and Q, U_n, $2U_n$, V_n or $2V_n$ is a square. In [31], Mignotte and Pethö showed that if $P \geq 3$ and $Q = -1$, then the equation $U_n = x^2$ has the solutions $(P,n) = (338,4)$ or $(3,6)$ for $n \geq 3$, and that if $P \geq 4$ and $Q = -1$, then the equation $U_n = wx^2$, $w \in \{2,3,6\}$, has no solutions for $n \geq 4$. In [34], Nakamula and Pethö have given the solutions of the equations $U_n = wx^2$ for $Q = 1$ with $w \in \{1,2,3,6\}$. In [40], Ribenboim and McDaniel showed that if P is even, $Q \equiv 3(mod 4)$, and $U_n = x^2$, then n is a square or twice an odd square, and all prime factors of n divide $P^2 + 4Q$. Also, in [42], they determined all indices such that for all odd relatively prime integers P and Q, $U_n = kx^2$ under the following assumptions: For all integer $u \geq 1$, k is such that, for each odd divisor h of k, the Jacobi symbol $\left(\frac{-V_{2^u}}{h}\right)$ is defined and equals to 1. Moreover, they solved the equation $V_n = 3x^2$ for $P \equiv 1,3(mod 8)$, $Q \equiv 3(mod 4)$, $(P,Q) = 1$ and solved the equation $U_n = 3x^2$ for all odd relatively prime integers P and Q. In [19], Kagawa and Terai showed that if $P = 2s$ with even s and $Q = 1$, then $U_n, 2U_n, V_n$ or $2V_n = x^2$ implies $n \leq 3$ under some assumptions.

To solve the equations mentioned above, divisibility properties, congruences, and Jacobi symbol were widely used by Cohn, Ribenboim and McDaniel.

In the second chapter of this thesis, some identities and summation formulas containing generalized Fibonacci and Lucas numbers are obtained. In finding these identities and summation formulas, generalized Fibonacci matrix $\begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix}$ and also the matrix $\begin{bmatrix} P/2 & (P^2+4Q)/2 \\ 1/2 & P/2 \end{bmatrix}$ are used. Using some of these identities and summation formulas, some congruences concerning generalized Fibonacci and Lucas numbers such as

$$V_{2mn+r} \equiv (-(-Q)^m)^n V_r \left(mod V_m\right), U_{2mn+r} \equiv (-(-Q)^m)^n U_r (mod V_m),$$

and

$$V_{2mn+r} \equiv (-Q)^{mn} V_r \,(modU_m),\, U_{2mn+r} \equiv (-Q)^{mn} U_r \,(modU_m),$$

are given. The matrices $\begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} P/2 & (P^2+4Q)/2 \\ 1/2 & P/2 \end{bmatrix}$ satisfy the characteristic equation $x^2 - Px - Q = 0$. All the 2×2 matrices $\mathbf{X}$ satisfying the relation $\mathbf{X}^2 = P\mathbf{X} + Q\mathbf{I}$ are also characterized in the second chapter. In the third chapter, the Diophantine equations $L_n = 2L_m x^2$, $F_n = 2F_m x^2$, $F_n = 3F_m x^2$, $L_n = 6L_m x^2$, and $F_n = 6F_m x^2$ are solved. Finally, in the fourth chapter, generalized Fibonacci and Lucas numbers of the form cx^2 are determined under some assumptions. The Jacobi symbol, the above congruences and divisibility properties are widely used in the solutions of the problems under consideration.

CHAPTER 2. SOME NEW IDENTITIES CONCERNING GENERALIZED FIBONACCI AND LUCAS NUMBERS

In this chapter, some identities containing generalized Fibonacci and Lucas numbers are obtained. Some of them are new and some are well known. Using these identities, some congruences concerning generalized Fibonacci and Lucas numbers are given.

Many identities concerning generalized Fibonacci and Lucas numbers can be proved using Binet's formulas, induction, and matrix representations. In the literature, for example in [14] and [20], the matrices

$$\begin{bmatrix} 0 & 1 \\ Q & P \end{bmatrix} \text{ and } \begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix}$$

are used in order to produce identities. Since

$$\begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ Q & P \end{bmatrix}$$

are similar matrices, they give the same identities.

In this chapter, we also characterize all the 2×2 matrices $\mathbf{X}$ satisfying the relation $\mathbf{X}^2 = P\mathbf{X} + Q\mathbf{I}$. Then some identities are obtained using this property. In fact, the similar matrices

$$\begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ Q & P \end{bmatrix}$$

are special cases of the 2×2 matrices $\mathbf{X}$ satisfying $\mathbf{X}^2 = P\mathbf{X} + Q\mathbf{I}$.

Theorem 2.1. If $\mathbf{X}$ is a square matrix with $\mathbf{X}^2 = P\mathbf{X} + Q\mathbf{I}$, then $\mathbf{X}^n = U_n\mathbf{X} + QU_{n-1}\mathbf{I}$ for every $n \in \mathbf{Z}$.

Proof: If $n = 0$, then the proof is obvious. It can be shown by induction that $\mathbf{X}^n = U_n\mathbf{X} + QU_{n-1}\mathbf{I}$ for every $n \in \mathbf{N}$. We now show that $\mathbf{X}^{-n} = U_{-n}\mathbf{X} + QU_{-n-1}\mathbf{I}$ for every $n \in \mathbf{N}$. Let $\mathbf{Y} = P\mathbf{I} - \mathbf{X} = -Q\mathbf{X}^{-1}$. Then

$$\begin{aligned}\mathbf{Y}^2 &= (P\mathbf{I} - \mathbf{X})^2 = P^2\mathbf{I} - 2P\mathbf{X} + \mathbf{X}^2 \\ &= P^2\mathbf{I} - 2P\mathbf{X} + P\mathbf{X} + Q\mathbf{I} = P(P\mathbf{I} - \mathbf{X}) + Q\mathbf{I} = P\mathbf{Y} + Q\mathbf{I}.\end{aligned}$$

Thus $\mathbf{Y}^n = U_n\mathbf{Y} + QU_{n-1}\mathbf{I}$ for every $n \in \mathbf{N}$, which shows that

$$\begin{aligned}(-Q)^n\mathbf{X}^{-n} &= U_n\mathbf{Y} + QU_{n-1}\mathbf{I} = U_n(P\mathbf{I} - \mathbf{X}) + QU_{n-1}\mathbf{I} \\ &= (PU_n + QU_{n-1})\mathbf{I} - U_n\mathbf{X} = -U_n\mathbf{X} + U_{n+1}\mathbf{I}.\end{aligned}$$

Then we get $\mathbf{X}^{-n} = \dfrac{-U_n\mathbf{X}}{(-Q)^n} + \dfrac{U_{n+1}\mathbf{I}}{(-Q)^n}$. This implies that $\mathbf{X}^{-n} = U_{-n}\mathbf{X} + QU_{-n-1}\mathbf{I}$ by (1.1). This completes the proof.

Theorem 2.2. Let $\mathbf{X}$ be an arbitrary 2×2 matrix. Then $\mathbf{X}^2 = P\mathbf{X} + Q\mathbf{I}$ if and only if $\mathbf{X}$ is of the form

$$\mathbf{X} = \begin{bmatrix} a & b \\ c & P-a \end{bmatrix}$$

for $a, b, c \in \mathbf{R}$ with $\det\mathbf{X} = -Q$ or $\mathbf{X} = \lambda\mathbf{I}$, where $\lambda \in \{\alpha, \beta\}$, $\alpha = \left(P + \sqrt{P^2 + 4Q}\right)/2$ and $\beta = \left(P - \sqrt{P^2 + 4Q}\right)/2$.

Proof: Assume that $\mathbf{X}^2 = P\mathbf{X} + Q\mathbf{I}$. Then the minimal polynomial of $\mathbf{X}$ divides $x^2 - Px - Q$. Therefore the minimal polynomial must be $x - \alpha$ or $x - \beta$ or $x^2 - Px - Q$. In the first case $\mathbf{X} = \alpha\mathbf{I}$, in the second case $\mathbf{X} = \beta\mathbf{I}$, and in the third case, since $\mathbf{X}$ is 2×2 matrix, its characteristic polynomial must be $x^2 - Px - Q$, so its trace is P and its determinant is $-Q$. The argument reverses. This completes the proof.

Corollary 2.1. If $\mathbf{X} = \begin{bmatrix} a & b \\ c & P-a \end{bmatrix}$ is a matrix with $\det\mathbf{X} = -Q$, then

$$\mathbf{X}^n = \begin{bmatrix} aU_n + QU_{n-1} & bU_n \\ cU_n & U_{n+1} - aU_n \end{bmatrix}.$$

Proof: Since $\mathbf{X}^2 = P\mathbf{X} + Q\mathbf{I}$, the result follows from Theorem 2.1.

Corollary 2.2. $\alpha^n = \alpha U_n + QU_{n-1}$ and $\beta^n = \beta U_n + QU_{n-1}$ for every $n \in \mathbf{Z}$.

Proof: Take $\mathbf{X} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ with $\det \mathbf{X} = \alpha\beta = -Q$. Then by Corollary 2.1, it follows that

$$\mathbf{X}^n = \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha U_n + QU_{n-1} & 0 \\ 0 & \beta U_n + QU_{n-1} \end{bmatrix}.$$

This implies that $\alpha^n = \alpha U_n + QU_{n-1}$ and $\beta^n = \beta U_n + QU_{n-1}$.

Corollary 2.3. $U_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n = \alpha^n + \beta^n$ for every $n \in \mathbf{Z}$.

Proof: The result follows from Corollary 2.2.

Corollary 2.4. Let $\mathbf{S} = \begin{bmatrix} P/2 & (P^2 + 4Q)/2 \\ 1/2 & P/2 \end{bmatrix}$. Then $\mathbf{S}^n = \begin{bmatrix} V_n/2 & (P^2 + 4Q)U_n/2 \\ U_n/2 & V_n/2 \end{bmatrix}$ for every $n \in \mathbf{Z}$.

Proof: Since $\mathbf{S}^2 = P\mathbf{S} + Q\mathbf{I}$, the proof follows from Corollary 2.1.

Corollary 2.5. Let $\mathbf{X} = \begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix}$. Then $\mathbf{X}^n = \begin{bmatrix} U_{n+1} & QU_n \\ U_n & QU_{n-1} \end{bmatrix}$.

Proof: Since $\mathbf{X}^2 = P\mathbf{X} + Q\mathbf{I}$, the proof follows from Corollary 2.1.

Lemma 2.1. Let a, b, and $Pa + b$ be nonzero real numbers. If $P^2 + 4Q$ is not a perfect square, then

$$\sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} U_{j+r} = -(-Q)^r \sum_{j=0}^{n} \binom{n}{j} (-a)^j (Pa + b)^{n-j} U_{j-r}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}V_{j+r} = (-Q)^r \sum_{j=0}^{n}\binom{n}{j}(-a)^j (Pa+b)^{n-j}V_{j-r} .$$

Proof: Let $\mathbf{Z}[\alpha]=\{a\alpha+b \mid a,b\in \mathbf{Z}\}$. Define $\phi:\mathbf{Z}[\alpha]\to\mathbf{Z}[\alpha]$ by $\varphi(a\alpha+b) = a\beta+b = a(P-\alpha)+b = -a\alpha+Pa+b.$ Then it can be shown that φ is a ring homomorphism. Moreover, it can be shown that φ is injective. On the other hand, we get

$$\begin{aligned}-\alpha U_n + U_{n+1} &= -\alpha U_n + PU_n + QU_{n-1} = \varphi(\alpha U_n + QU_{n-1})\\ &= \varphi(\alpha^n) = \beta^n = (-Q)^n \alpha^{-n} .\end{aligned}$$

Then it is seen that

$$\begin{aligned}\varphi((a\alpha+b)^n\alpha^r) &= \varphi((a\alpha+b)^n)\varphi(\alpha^r) = (-a\alpha+Pa+b)^n(-Q)^r\alpha^{-r}\\
&= (-Q)^r\sum_{j=0}^{n}\binom{n}{j}(-a\alpha)^j(Pa+b)^{n-j}\alpha^{-r}\\
&= (-Q)^r\sum_{j=0}^{n}\binom{n}{j}(-a)^j(Pa+b)^{n-j}\alpha^{j-r}\\
&= (-Q)^r\sum_{j=0}^{n}\binom{n}{j}(-a)^j(Pa+b)^{n-j}(\alpha U_{j-r}+QU_{j-r-1})\\
&= \alpha\left((-Q)^r\sum_{j=0}^{n}\binom{n}{j}(-a)^j(Pa+b)^{n-j}U_{j-r}\right)\\
&\quad+\left(-(-Q)^{r+1}\sum_{j=0}^{n}\binom{n}{j}(-a)^j(Pa+b)^{n-j}U_{j-r-1}\right).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\varphi((a\alpha+b)^n\alpha^r) &= \varphi\left(\sum_{j=0}^{n}\binom{n}{j}a^jb^{n-j}\alpha^{j+r}\right)\\
&= \varphi\left(\sum_{j=0}^{n}\binom{n}{j}a^jb^{n-j}(\alpha U_{j+r}+QU_{j+r-1})\right)\\
&= \alpha\left(-\sum_{j=0}^{n}\binom{n}{j}a^jb^{n-j}U_{j+r}\right)\\
&\quad+\left(\sum_{j=0}^{n}\binom{n}{j}a^jb^{n-j}(PU_{j+r}+QU_{j+r-1})\right)\\
&= \alpha\left(-\sum_{j=0}^{n}\binom{n}{j}a^jb^{n-j}U_{j+r}\right)+\left(\sum_{j=0}^{n}\binom{n}{j}a^jb^{n-j}U_{j+r+1}\right).\end{aligned}$$

Then the proof follows.

Theorem 2.3. Let $m, r \in \mathbf{Z}$ with $m \neq 0$ and $m \neq 1$. Then

$$U_{mn+r} = \sum_{j=0}^{n} \binom{n}{j} U_m^j U_{m-1}^{n-j} U_{j+r} Q^{n-j}$$

and

$$V_{mn+r} = \sum_{j=0}^{n} \binom{n}{j} U_m^j U_{m-1}^{n-j} V_{j+r} Q^{n-j} .$$

Proof: From Corollary 2.4, it follows that

$$\mathbf{S}^{mn+r} = \begin{bmatrix} \frac{V_{mn+r}}{2} & \frac{(P^2+4Q)U_{mn+r}}{2} \\ \frac{U_{mn+r}}{2} & \frac{V_{mn+r}}{2} \end{bmatrix}.$$

On the other hand, $\mathbf{S}^m = U_m \mathbf{S} + QU_{m-1}\mathbf{I}$ and therefore

$$\mathbf{S}^{mn+r} = (\mathbf{S}^m)^n \mathbf{S}^r = (U_m \mathbf{S} + QU_{m-1}\mathbf{I})^n \mathbf{S}^r = \sum_{j=0}^{n} \binom{n}{j} U_m^j U_{m-1}^{n-j} Q^{n-j} \mathbf{S}^{j+r}$$

$$= \begin{bmatrix} \frac{1}{2}\sum_{j=0}^{n} \binom{n}{j} U_m^j U_{m-1}^{n-j} Q^{n-j} V_{j+r} & \frac{(P^2+4Q)}{2}\sum_{j=0}^{n} \binom{n}{j} U_m^j U_{m-1}^{n-j} Q^{n-j} U_{j+r} \\ \frac{1}{2}\sum_{j=0}^{n} \binom{n}{j} U_m^j U_{m-1}^{n-j} Q^{n-j} U_{j+r} & \frac{1}{2}\sum_{j=0}^{n} \binom{n}{j} U_m^j U_{m-1}^{n-j} Q^{n-j} V_{j+r} \end{bmatrix}.$$

So, the proof follows.

Corollary 2.6. Let $m, r \in \mathbf{Z}$ with $m \neq 0$ and $m \neq 1$. If $P^2 + 4Q$ is not a perfect square, then

$$U_{mn+r} = -(-Q)^r \sum_{j=0}^{n} \binom{n}{j} (-U_m)^j U_{m+1}^{n-j} U_{j-r}$$

and

$$V_{mn+r} = (-Q)^r \sum_{j=0}^{n} \binom{n}{j} (-U_m)^j U_{m+1}^{n-j} V_{j-r} .$$

Proof: The proof follows from Lemma 2.1 and Theorem 2.3 by taking $a = U_m$ and $b = QU_{m-1}$.

Corollary 2.7. $V_n^2 - (P^2+4Q)U_n^2 = 4(-Q)^n$ for every $n \in \mathbf{Z}$.

Proof: From Corollary 2.4, it follows that

$$\det \mathbf{S}^n = (\det \mathbf{S})^n = (-Q)^n$$

and

$$\det \mathbf{S}^n = \frac{V_n^2 - (P^2+4Q)U_n^2}{4},$$

which completes the proof.

Theorem 2.4. Let $n \in \mathbf{N}$ and m be a nonzero integer. Then

$$2^n V_{mn+r} = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} U_m^{2j} V_m^{n-2j} (P^2+4Q)^j V_r + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} U_m^{2j+1} V_m^{n-2j-1} (P^2+4Q)^{j+1} U_r$$

and

$$2^n U_{mn+r} = \frac{1}{2^n} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} U_m^{2j} V_m^{n-2j} (P^2+4Q)^j U_r + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} U_m^{2j+1} V_m^{n-2j-1} (P^2+4Q)^j V_r .$$

Proof: Let $\mathbf{K} = \mathbf{S} + Q\mathbf{S}^{-1} = \begin{bmatrix} 0 & P^2+4Q \\ 1 & 0 \end{bmatrix}$. Then $\mathbf{K}^{2j} = (P^2+4Q)^j \mathbf{I}$ and $\mathbf{K}^{2j+1} = (P^2+4Q)^j \mathbf{K}$. Since

$$\mathbf{S}^m = \frac{1}{2}(V_m \mathbf{I} + U_m \mathbf{K}),$$

it follows that

$$\mathbf{S}^{mn+r} = (\mathbf{S}^m)^n \mathbf{S}^r = \left(\frac{1}{2}(V_m \mathbf{I} + U_m \mathbf{K})\right)^n \mathbf{S}^r = \frac{1}{2^n}\left(\sum_{j=0}^{n} \binom{n}{j} U_m^j K^j V_m^{n-j}\right)\mathbf{S}^r$$

and therefore

$$2^n\mathbf{S}^{mn+r} = \sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n}{2j}U_m^{2j}V_m^{n-2j}\mathbf{K}^{2j}\mathbf{S}^r + \sum_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n}{2j+1}U_m^{2j+1}V_m^{n-2j-1}\mathbf{K}^{2j+1}\mathbf{S}^r$$
$$= \sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n}{2j}U_m^{2j}V_m^{n-2j}(P^2+4Q)^j\mathbf{S}^r$$
$$+ \sum_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n}{2j+1}U_m^{2j+1}V_m^{n-2j-1}(P^2+4Q)^j\mathbf{KS}^r .$$

Since

$$\mathbf{KS}^r = \begin{bmatrix} \frac{(P^2+4Q)U_r}{2} & \frac{(P^2+4Q)V_r}{2} \\ \frac{V_r}{2} & \frac{(P^2+4Q)U_r}{2} \end{bmatrix}$$

and

$$\mathbf{S}^{mn+r} = \begin{bmatrix} \frac{V_{mn+r}}{2} & \frac{(P^2+4Q)U_{mn+r}}{2} \\ \frac{U_{mn+r}}{2} & \frac{V_{mn+r}}{2} \end{bmatrix},$$

the proof is completed.

Theorem 2.5.

$$U_{m+n} = U_mU_{n+1} + QU_{m-1}U_n \tag{2.1}$$

and

$$(-Q)^{n-1}U_{m-n} = U_{m-1}U_n - U_mU_{n-1} \tag{2.2}$$

for every $m, n \in \mathbf{Z}$.

Proof: Let $\mathbf{X} = \begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix}$. Then from Corollary 2.5, it follows that

$$\mathbf{X}^{m+n} = \mathbf{X}^m\mathbf{X}^n = \begin{bmatrix} U_{m+1} & QU_m \\ U_m & QU_{m-1} \end{bmatrix}\begin{bmatrix} U_{n+1} & QU_n \\ U_n & QU_{n-1} \end{bmatrix}$$

and

$$\mathbf{X}^{m-n} = \mathbf{X}^m(\mathbf{X}^n)^{-1} = \begin{bmatrix} U_{m+1} & QU_m \\ U_m & QU_{m-1} \end{bmatrix} \begin{bmatrix} U_{n+1} & QU_n \\ U_n & QU_{n-1} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} U_{m+1} & QU_m \\ U_m & QU_{m-1} \end{bmatrix} \frac{1}{(-Q)^n} \begin{bmatrix} QU_{n-1} & -QU_n \\ -U_n & U_{n+1} \end{bmatrix}.$$

Then the proof is completed.

Now some identities are given which will be used in the sequel. These identities can be obtained using matrices $\mathbf{S}^n$ and $\mathbf{X}^n$ and they are;

$$U_n V_{m+1} + QU_{n-1} V_m = V_{n+m}, \tag{2.3}$$

$$V_m V_n - (P^2 + 4Q) U_m U_n = 2(-Q)^n V_{m-n}, \tag{2.4}$$

$$U_m V_n - U_n V_m = 2(-Q)^n U_{m-n}, \tag{2.5}$$

$$V_m V_n = V_{m+n} + (-Q)^n V_{m-n}, \tag{2.6}$$

$$(P^2 + 4Q) U_m U_n = V_{m+n} - (-Q)^n V_{m-n}, \tag{2.7}$$

$$U_m V_n = U_{m+n} + (-Q)^n U_{m-n}, \tag{2.8}$$

$$(-Q)^n V_{m-n} = U_{m+1} V_n - V_{n+1} U_m, \tag{2.9}$$

$$V_r V_{r+2} - V_{r+1}^2 = (-Q)^r (P^2 + 4Q). \tag{2.10}$$

Theorem 2.6. Let $m, n, r \in \mathbf{Z}$ with $r \neq 0$. Then

$$U_r U_{m+n+r} = U_{m+r} U_{n+r} - (-Q)^r U_m U_n,$$

$$U_r U_{m+n-r} = U_m U_n - (-Q)^r U_{m-r} U_{n-r},$$

and

$$U_r U_{m+n} = U_m U_{n+r} - (-Q)^r U_{m-r} U_n.$$

Proof: Take $a = \dfrac{U_{r+1}}{U_r}$ and consider the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & P-a \end{bmatrix}$ with $\det \mathbf{A} = -Q$.

Then by Corollary 2.1, one gets

$$\mathbf{A}^n = \begin{bmatrix} aU_n + QU_{n-1} & bU_n \\ cU_n & U_{n+1} - aU_n \end{bmatrix} = \begin{bmatrix} \dfrac{U_{r+1}}{U_r} U_n + QU_{n-1} & bU_n \\ cU_n & U_{n+1} - \dfrac{U_{r+1}}{U_r} U_n \end{bmatrix}.$$

Using (2.1) and (2.2), it is seen that

$$\mathbf{A}^n = \begin{bmatrix} \dfrac{U_{n+r}}{U_r} & bU_n \\ cU_n & \dfrac{-(-Q)^r U_{n-r}}{U_r} \end{bmatrix}.$$

Since $\det \mathbf{A} = -Q$ and $a = \dfrac{U_{r+1}}{U_r}$, it follows that

$$bc = \frac{PU_rU_{r+1} + QU_r^2 - U_{r+1}^2}{U_r^2} = \frac{U_r(PU_{r+1} + QU_r) - U_{r+1}^2}{U_r^2}$$
$$= \frac{U_rU_{r+2} - U_{r+1}^2}{U_r^2} = \frac{-(-Q)^r}{U_r^2}$$

by (2.2). If we consider the matrix product $\mathbf{A}^n\mathbf{A}^m = \mathbf{A}^{m+n}$, then the result follows.

We can give the following corollary.

Corollary 2.8. $U_{n+r}U_{n-r} - U_n^2 = -(-Q)^{n-r}U_r^2$ for all $n, r \in \mathbf{Z}$.

Proof: Since $\det \mathbf{A} = -Q$, $\det \mathbf{A}^n = (\det \mathbf{A})^n = (-Q)^n$. Moreover,

$$\det \mathbf{A}^n = -(-Q)^r \frac{U_{n+r}}{U_r}\frac{U_{n-r}}{U_r} - bcU_n^2 = -(-Q)^r\left(\frac{U_{n+r}U_{n-r} - U_n^2}{U_r^2}\right),$$

implies that $U_{n+r}U_{n-r} - U_n^2 = -(-Q)^{n-r}U_r^2$.

Theorem 2.7. Let $m, n, r \in \mathbf{Z}$. Then

$$V_rV_{m+n+r} = V_{m+r}V_{n+r} + (-Q)^r(P^2 + 4Q)U_mU_n,$$
$$V_rV_{m+n-r} = (P^2 + 4Q)U_mU_n + (-Q)^rV_{m-r}V_{n-r},$$

and

$$V_rU_{m+n} = U_nV_{m+r} + (-Q)^rV_{n-r}U_m.$$

Proof: Take $a = \dfrac{V_{r+1}}{V_r}$ and consider the matrix $\mathbf{B} = \begin{bmatrix} a & b \\ c & P-a \end{bmatrix}$ with $\det \mathbf{B} = -Q$.

Then by Corollary 2.1, one gets

$$\mathbf{B}^n = \begin{bmatrix} aU_n + QU_{n-1} & bU_n \\ cU_n & U_{n+1} - aU_n \end{bmatrix} = \begin{bmatrix} \frac{V_{r+1}}{V_r}U_n + QU_{n-1} & bU_n \\ cU_n & U_{n+1} - \frac{V_{r+1}}{V_r}U_n \end{bmatrix}.$$

Hence, using (2.3) and (2.9), it is seen that

$$\mathbf{B}^n = \begin{bmatrix} \frac{V_{n+r}}{V_r} & bU_n \\ cU_n & \frac{(-Q)^r V_{n-r}}{V_r} \end{bmatrix}.$$

Since $\det \mathbf{B} = -Q$ and $a = \frac{V_{r+1}}{V_r}$, it follows that

$$bc = \frac{PV_rV_{r+1} + QV_r^2 - V_{r+1}^2}{V_r^2} = \frac{V_r(PV_{r+1} + QV_r) - V_{r+1}^2}{V_r^2}$$
$$= \frac{V_rV_{r+2} - V_{r+1}^2}{V_r^2} = \frac{(-Q)^r(P^2 + 4Q)}{V_r^2}$$

by (2.10). Then we get the result by considering the matrix product $\mathbf{B}^n\mathbf{B}^m = \mathbf{B}^{m+n}$.

Now, the following corollary can be given.

Corollary 2.9. $V_{n+r}V_{n-r} - (P^2 + 4Q)U_n^2 = (-Q)^{n-r}V_r^2$ for all $n, r \in \mathbf{Z}$.

Proof: Since $\det \mathbf{B} = -Q$, $\det \mathbf{B}^n = (\det \mathbf{B})^n = (-Q)^n$ and

$$\det \mathbf{B}^n = (-Q)^r \frac{V_{n+r}}{V_r}\frac{V_{n-r}}{V_r} - bcU_n^2 = (-Q)^r\left(\frac{V_{n+r}V_{n-r}}{V_r^2} - \frac{(P^2 + 4Q)U_n^2}{V_r^2}\right),$$

it follows that $V_{n+r}V_{n-r} - (P^2 + 4Q)U_n^2 = (-Q)^{n-r}V_r^2$.

2.1. Sums and Congruences

Now some sums containing generalized Fibonacci and Lucas numbers will be given. Then, some congruences concerning generalized Fibonacci and Lucas numbers will be presented. Before giving a lemma which will be used in the theorems following it, notice that

$$\alpha^{2n} = \alpha^n V_n - (-Q)^n \tag{2.11}$$

and

$$\alpha^{2n} = \alpha^n U_n \sqrt{P^2+4Q} + (-Q)^n \qquad (2.12)$$

by (1.2). Now we can give the lemma.

Lemma 2.1.1.

$$\mathbf{S}^{2n} = V_n \mathbf{S}^n - (-Q)^n \mathbf{I} \qquad (2.13)$$

and

$$\mathbf{S}^{2n} = U_n \mathbf{K}\mathbf{S}^n + (-Q)^n \mathbf{I} \qquad (2.14)$$

for every $n \in \mathbf{N}$, where $\mathbf{K}$ is as in Theorem 2.4.

Proof: Let $\mathbf{Z}[\alpha] = \{a\alpha + b \mid a, b \in \mathbf{Z}\}$, $\mathbf{Z}[\mathbf{S}] = \{a\mathbf{S} + b\mathbf{I} \mid a, b \in \mathbf{Z}\}$ and define a function $\phi : \mathbf{Z}[\alpha] \to \mathbf{Z}[\mathbf{S}]$, by $\varphi(a\alpha + b) = a\mathbf{S} + b\mathbf{I}$. Then φ is a ring homomorphism. Moreover, it is clear that $\varphi(\alpha) = \mathbf{S}$ and therefore $\varphi(\alpha^n) = (\varphi(\alpha))^n = \mathbf{S}^n$. Thus from (2.11), one gets

$$\mathbf{S}^{2n} = (\varphi(\alpha))^{2n} = \varphi(\alpha^{2n}) = \varphi(\alpha^n V_n - (-Q)^n) = V_n \mathbf{S}^n - (-Q)^n \mathbf{I}.$$

That is, $\mathbf{S}^{2n} = V_n \mathbf{S}^n - (-Q)^n \mathbf{I}$. Also from (2.12), it follows that

$$\mathbf{S}^{2n} = (\varphi(\alpha))^{2n} = \varphi(\alpha^{2n}) = \varphi\left(U_n \sqrt{P^2+4Q}\,\alpha^n + (-Q)^n\right) = U_n \varphi\left(\sqrt{P^2+4Q}\right)\mathbf{S}^n + (-Q)^n \mathbf{I}.$$

Then $\mathbf{S}^{2n} = U_n \mathbf{K}\mathbf{S}^n + (-Q)^n \mathbf{I}$ since

$$\varphi\left(\sqrt{P^2+4Q}\right) = \varphi(2\alpha - P) = 2\mathbf{S} - P\mathbf{I} = \begin{bmatrix} 0 & P^2+4Q \\ 1 & 0 \end{bmatrix} = \mathbf{K}.$$

Theorem 2.1.1. Let $m, r \in \mathbf{Z}$. Then

$$U_{2mn+r} = (-(-Q)^m)^n \sum_{j=0}^{n} \binom{n}{j} V_m^j U_{mj+r} (-(-Q)^m)^{-j}$$

and

$$V_{2mn+r} = (-(-Q)^m)^n \sum_{j=0}^{n} \binom{n}{j} V_m^j V_{mj+r} (-(-Q)^m)^{-j}$$

for every $n \in \mathbf{N}$.

Proof: It is known that

$$\mathbf{S}^{2m} = V_m\mathbf{S}^m - (-Q)^m\mathbf{I} \tag{2.15}$$

by (2.13). Taking the nth power of (2.15), one gets

$$\mathbf{S}^{2mn} = \left(V_m\mathbf{S}^m - (-Q)^m\mathbf{I}\right)^n = \sum_{j=0}^{n}\binom{n}{j}V_m^j(-(-Q)^m)^{n-j}\mathbf{S}^{mj}.$$

Multiplication of both sides of this equation by $\mathbf{S}^r$ gives

$$\mathbf{S}^{2mn+r} = (-(-Q)^m)^n\sum_{j=0}^{n}\binom{n}{j}V_m^j(-(-Q)^m)^{-j}\mathbf{S}^{mj+r}.$$

Thus it follows that

$$U_{2mn+r} = (-(-Q)^m)^n\sum_{j=0}^{n}\binom{n}{j}V_m^jU_{mj+r}(-(-Q)^m)^{-j}$$

and

$$V_{2mn+r} = (-(-Q)^m)^n\sum_{j=0}^{n}\binom{n}{j}V_m^jV_{mj+r}(-(-Q)^m)^{-j}$$

by Corollary 2.4.

Corollary 2.1.1. If P and Q are integers, then

$$U_{2mn+r} \equiv (-(-Q)^m)^nU_r\left(modV_m\right) \tag{2.16}$$

and

$$V_{2mn+r} \equiv (-(-Q)^m)^nV_r\left(modV_m\right) \tag{2.17}$$

for all $n, m \in \mathbf{N}\cup\{0\}$ and $r \in \mathbf{Z}$ such that $mn + r \geq 0$ if $Q \neq \pm 1$.

Theorem 2.1.2. Let $m, r \in \mathbf{Z}$ and m be a nonzero integer. Then

$$U_{2mn+r} = (-Q)^{mn}\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n}{2j}U_m^{2j}U_{2mj+r}D^jQ^{-2mj}$$
$$+ (-Q)^{mn}\sum_{j=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n}{2j+1}U_m^{2j+1}V_{2mj+m+r}D^j(-Q)^{m(-2j-1)}$$

and

$$V_{2mn+r} = (-Q)^{mn} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} U_m^{2j} V_{2mj+r} D^j Q^{-2mj}$$
$$+ (-Q)^{mn} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} U_m^{2j+1} U_{2mj+m+r} D^{j+1} (-Q)^{m(-2j-1)}$$

for every $n \in \mathbf{N}$, where $D = P^2 + 4Q.$

Proof: It is known that

$$\mathbf{S}^{2m} = U_m \mathbf{K}\mathbf{S}^m + (-Q)^m \mathbf{I}$$

by (2.14). Then,

$$\mathbf{S}^{2mn+r} = \left(U_m \mathbf{K}\mathbf{S}^m + (-Q)^m \mathbf{I}\right)^n \mathbf{S}^r = \sum_{j=0}^{n} \binom{n}{j} U_m^j \mathbf{K}^j ((-Q)^m)^{n-j} \mathbf{S}^{mj+r}.$$

On the other hand, it can be seen that $\mathbf{K}^{2j} = D^j \mathbf{I}$ and $\mathbf{K}^{2j+1} = D^j \mathbf{K}.$ Therefore,

$$\mathbf{S}^{2mn+r} = (-Q)^{mn} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} U_m^{2j} \mathbf{K}^{2j} Q^{-2mj} \mathbf{S}^{2mj+r}$$
$$+ (-Q)^{mn} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} U_m^{2j+1} \mathbf{K}^{2j+1} (-Q)^{m(-2j-1)} \mathbf{S}^{2mj+m+r}$$
$$= (-Q)^{mn} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} U_m^{2j} D^j Q^{-2mj} \mathbf{S}^{2mj+r}$$
$$+ (-Q)^{mn} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} U_m^{2j+1} D^j (-Q)^{m(-2j-1)} \mathbf{K}\mathbf{S}^{2mj+m+r} ,$$

and the proof follows from Corollary 2.4.

Corollary 2.1.2. If P and Q are integers, then

$$U_{2mn+r} \equiv (-Q)^{mn} U_r \left(mod U_m\right) \tag{2.18}$$

and

$$V_{2mn+r} \equiv (-Q)^{mn} V_r \left(mod U_m\right) \tag{2.19}$$

for all $n, m \in \mathbf{N} \cup \{0\}$ and $r \in \mathbf{Z}$ such that $mn + r \geq 0$ if $Q \neq \pm 1$.

CHAPTER 3. THE SQUARE TERMS IN FIBONACCI AND LUCAS SEQUENCES

We have cited literature review about Fibonacci and Lucas numbers of the form x^2 or $2x^2$ in the first chapter. Many authors have investigated Fibonacci and Lucas numbers of the form cx^2 with $c \neq 1,2$. For example, in [43], Robbins considered Fibonacci numbers of the form px^2 and solved the equation $F_n = px^2$ for all p such that $p \equiv 3(mod 4)$ or $p < 10000$. On the other hand, in [44], Robbins considered Fibonacci numbers of the form cx^2 and obtained all solutions of $F_n = cx^2$ for composite values of $c \leq 1000$. The same author solved $L_n = px^2$, where p is an odd prime and $p < 1000$, in [45]. Moreover, in [52], Zhou dealt with Lucas numbers of the form px^2, where p is a prime number and gave solutions for $1000 < p < 60000$.

In this chapter, firstly, some fundamental theorems and identities concerning Fibonacci and Lucas numbers are given. Then, we solve the Diophantine equations $L_n = 2L_m x^2$, $F_n = 2F_m x^2$, $F_n = 3F_m x^2$, $L_n = 6L_m x^2$, and $F_n = 6F_m x^2$.

The proof of the following theorem can be found in [7].

Theorem 3.1. Let $n \geq 1$. If $F_n = x^2$, then $n = 1, 2, 12$. If $F_n = 2x^2$, then $n = 3, 6$. If $L_n = x^2$, then $n = 1, 3$ and if $L_n = 2x^2$, then $n = 6$.

The proofs of the following two theorems are given in [22].

Theorem 3.2. Let $n \in \mathbf{N} \cup \{0\}$ and $k, m \in \mathbf{Z}$. Then

$$F_{2mn+k} \equiv (-1)^{mn} F_k \left(mod F_m\right) \tag{3.1}$$

and

$$L_{2mn+k} \equiv (-1)^{mn} L_k \left(mod F_m \right). \tag{3.2}$$

Theorem 3.3. Let $n \in \mathbf{N} \cup \{0\}$ and $k, m \in \mathbf{Z}$. Then

$$L_{2mn+k} \equiv (-1)^{(m+1)n} L_k \left(mod L_m \right) \tag{3.3}$$

and

$$F_{2mn+k} \equiv (-1)^{(m+1)n} F_k \left(mod L_m \right). \tag{3.4}$$

The two theorems given above can be obtained from Corollary 2.1.1 and Corollary 2.1.2. From the identity (3.2), it can be easily seen that $8 \nmid L_n$ for every $n \in \mathbf{N}$.

The proofs of the following three lemmas can be done by induction.

Lemma 3.1. $L_{2^k} \equiv 3(mod\, 4)$ for all $k \geq 1$.

Lemma 3.2. If $r \geq 3$, then $L_{2^r} \equiv 2(mod\, 3)$.

Lemma 3.3. If $r \geq 2$, then $L_{2^r} \equiv 7(mod\, 8)$.

The following corollary can be obtained from Lemma 3.1.

Corollary 3.1. If $k \geq 1$, then there is no integer x such that $x^2 \equiv -1(mod L_{2^k})$.

The proofs of the following theorems can be found in [5], [22], and [48].

Theorem 3.4. Let $m, n \in \mathbf{N}$ and $m \geq 2$. Then $L_m \mid L_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is an odd integer.

Theorem 3.5. Let $m, n \in \mathbf{N}$ and $m \geq 3$. Then $F_m \mid F_n$ if and only if $m \mid n$.

Theorem 3.6. Let $m, n \in \mathbf{N}$ and $m \geq 2$. Then $L_m \mid F_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is an even integer.

The following identities are well known and easy to show.

$$L_{2n} = L_n^2 - 2(-1)^n, \tag{3.5}$$

$$L_{3n} = L_n(L_n^2 - 3(-1)^n), \tag{3.6}$$

$$F_{2n} = F_n L_n, \tag{3.7}$$

$$F_{3n} = F_n(5F_n^2 + 3(-1)^n), \tag{3.8}$$

$$L_n^2 - 5F_n^2 = 4(-1)^n, \tag{3.9}$$

$$2 \mid F_n \Leftrightarrow 2 \mid L_n \Leftrightarrow 3 \mid n, \tag{3.10}$$

$$(F_n, L_n) = 1 \text{ or } (F_n, L_n) = 2. \tag{3.11}$$

Let $\left(\frac{a}{p}\right)$ represent Legendre symbol. Then we have

$$\left(\frac{2}{p}\right) = 1 \text{ if and only if } p \equiv \mp 1 (mod 8) \tag{3.12}$$

and

$$\left(\frac{-2}{p}\right) = 1 \text{ if and only if } p \equiv 1, 3 (mod 8). \tag{3.13}$$

The proofs of (3.12) and (3.13) can be found in [4] and [35].

3.1. Fibonacci and Lucas Numbers of The Form cx^2

In this subsection, we consider the equations $L_n = 2L_m x^2$, $F_n = 2F_m x^2$, $L_n = 6L_m x^2$, $F_n = 3F_m x^2$, and $F_n = 6F_m x^2$.

In [38], Ribenboim introduced square-classes of Fibonacci numbers. There it is stated that F_m, F_n are in the same square-class if there exist non-zero integers *x, y* such that $F_m x^2 = F_n y^2$; or equivalently, when $F_m F_n$ is a square. In a similar way, he introduced square-classes of Lucas numbers. A square-class is called trivial if it consists of only one number. Ribenboim showed that the square-class of L_m is trivial when $m \neq 0, 1, 3,$ and $6.$ Also he showed that the square-class of F_m is trivial when $m \neq 1, 2, 3, 6, 12.$ Now, the following two theorems, which can be obtained from Proposition 1 and Proposition 2 in [38], can be given.

Throughout this subsection, we will assume that n is a positive integer.

Theorem 3.1.1. Let $m > 3$ be an integer and $F_n = F_m x^2$ for some $x \in \mathbf{Z}$. Then $n = m$.

Theorem 3.1.2. Let $m \geq 2$ be an integer and $L_n = L_m x^2$ for some $x \in \mathbf{Z}$. Then $n = m$.

Although the proofs of the following two theorems can be obtained from Theorem 6 and Theorem 12 in [11], proofs will be given using a different approach.

Theorem 3.1.3. There is no integer x such that $L_n = 2L_m x^2$ for $m > 1$.

Proof: Assume that $L_n = 2L_m x^2$. Then $L_m \mid L_n$ and therefore $n = mk$ for some natural number k by Theorem 3.4. Firstly, assume that m is an odd integer. Since $2 \mid L_n$, one gets $3 \mid n$ by (3.10). Thus it is seen that $3 \nmid m$. For, if $3 \mid m$, then $L_3 \mid L_m$, i.e., $4 \mid L_m$ by Theorem 3.4. This implies that $8 \mid L_n$, which is impossible. Since $3 \nmid m$, it follows that $3 \mid k$. That is, $k = 3t$ for some odd positive integer t. Thus $n = mk = 3mt$ and mt is an odd integer. Therefore, since $3 \mid n$, it follows that $L_3 \mid L_n$, i.e., $4 \mid 2L_m x^2$ by Theorem 3.4. Since $3 \nmid m$, L_m is an odd integer. Therefore $2 \mid x^2$, i.e., x is an even integer. This implies that $8 \mid L_n$, which is impossible.

Now, assume that m is an even integer. If x is an even integer, then it is seen that $8 | L_n$, which is impossible. Therefore x is an odd integer. Assume that $3 | m.$ Then L_m is an even integer. Therefore $L_3 | L_n$ by Theorem 3.4. It follows that $n = 3b$ for some odd integer b. That is, n is an odd integer. But this is impossible. Because m is an even integer, n is also an even integer. Assume that $3 \nmid m$. Then since $n = mk$ and $3 | n,$ it follows that $3 | k,$ i.e., $k = 3t$ for some odd integer t. Hence, $t = 4q \mp 1$ for some nonnegative integer q. Thus $n = mk = 3m(4q \mp 1) = 2.6mq \mp 3m.$ Then

$$L_n = L_{2.6mq \mp 3m} \equiv L_{\mp 3m} (mod F_6)$$

and therefore

$$2L_m x^2 \equiv L_{3m} (mod 8)$$

by (3.2). Since $x^2 \equiv 1(mod 8)$ and m is an even integer, one obtains

$$2L_m \equiv L_m (L_m^2 - 3)(mod 8)$$

by (3.6). Moreover, since $3 \nmid m$, L_m is odd integer. So

$$2 \equiv L_m^2 - 3(mod 8).$$

Whence

$$2 \equiv -2(mod 8),$$

which is not possible. This completes the proof.

In [7], for $m = 1, 2$, it is shown that the equation $F_n = 2F_m x^2 = 2x^2$ has solutions only for $n = 3,6.$ More generally, the following theorem can be given.

Theorem 3.1.4. If $F_n = 2F_m x^2$ and $m \geq 3,$ then $m = 3,$ $x^2 = 36,$ and $n = 12$ or $m = 6,$ $x^2 = 9,$ and $n = 12.$

Proof: If $m = 3,$ then $F_n = 2F_3 x^2 = (2x)^2.$ Thus it can be seen that $n = 12,$ $x^2 = 36$ by Theorem 3.1. Assume that $m > 3$ and $F_n = 2F_m x^2.$ Then $F_m | F_n$ and therefore $n = mk$ for some natural number k by Theorem 3.5.

Firstly, assume that k is an even integer. Then $k = 2t$ for some integer t. Therefore $n = mk = 2mt$. Thus

$$F_n = F_{2mt} = F_{mt}L_{mt} = 2F_m x^2$$

by (3.7). This shows that $(F_{mt}/F_m)L_{mt} = 2x^2$. It can be easily seen that if $(F_{mt}/F_m, L_{mt}) = d$, then $d = 1$ or $d = 2$ by (3.11). Assume that $d = 1$. Then

$$\frac{F_{mt}}{F_m} = u^2,\ L_{mt} = 2v^2 \tag{3.14}$$

or

$$\frac{F_{mt}}{F_m} = 2u^2,\ L_{mt} = v^2 \tag{3.15}$$

for some integers u and v. Assume that (3.14) is satisfied. Then $mt = m$, i.e., $t = 1$ by Theorem 3.1.1. Therefore $L_m = 2v^2$ and this implies that $m = 6$ by Theorem 3.1. Hence $x^2 = 9$ and $n = 12$. Now assume that (3.15) is satisfied. Then $L_{mt} = v^2$ and therefore $mt = 1$ or 3 by Theorem 3.1. But this is not possible since $m > 3$. Assume that $d = 2$. Then

$$\frac{F_{mt}}{F_m} = 2u^2,\ L_{mt} = v^2 \tag{3.16}$$

or

$$\frac{F_{mt}}{F_m} = u^2,\ L_{mt} = 2v^2 \tag{3.17}$$

for some integers u and v. Assume that (3.16) is satisfied. Then $mt = 1$ or 3 by Theorem 3.1. But this is impossible since $m > 3$. It can be seen that the identity (3.17) is impossible by Theorem 3.1.1.

Now, assume that k is an odd integer. Let m be an even integer. Then $m = 2r$ for some natural number r and therefore $n = mk = 2kr$. Thus one has

$$F_n = F_{2kr} = F_{kr}L_{kr} = 2F_r L_r x^2$$

by (3.7). This shows that $(F_{kr}/F_r)(L_{kr}/L_r) = 2x^2$. It can be easily seen that if $(F_{kr}/F_r, L_{kr}/L_r) = d$, then $d = 1$ or $d = 2$ by (3.11). Assume that $d = 1$. Then

$$\frac{F_{kr}}{F_r} = u^2,\ \frac{L_{kr}}{L_r} = 2v^2 \tag{3.18}$$

or

$$\frac{F_{kr}}{F_r} = 2u^2, \ \frac{L_{kr}}{L_r} = v^2 \tag{3.19}$$

for some integers u and v. The identity (3.18) is impossible by Theorem 3.1.3. Assume that (3.19) is satisfied. Then $L_{kr} = L_r v^2$ and therefore $kr = r$, i.e., $k = 1$ by Theorem 3.1.2. Hence the equality $2u^2 = \frac{F_{kr}}{F_r} = \frac{F_r}{F_r} = 1$ is obtained, which is not possible. Assume that $d = 2.$ Then

$$\frac{F_{kr}}{F_r} = 2u^2, \ L_{kr} = v^2 \tag{3.20}$$

or

$$\frac{F_{kr}}{F_r} = u^2, \ \frac{L_{kr}}{L_r} = 2v^2 \tag{3.21}$$

for some integers u and v. A similar argument shows that (3.20) and (3.21) are impossible. Now, let m be an odd integer. Firstly, suppose that $3 \nmid k$. Since k is an odd integer, $k = 6q \mp 1$ for some nonnegative integer q. Therefore $n = mk = m(6q \mp 1) = 2.3mq \mp m.$ Hence it follows that

$$F_n = F_{2.3mq \mp m} \equiv F_{\mp m} (mod L_3),$$

i.e.,

$$F_n \equiv F_m (mod\, 4)$$

by (3.4). Since F_n is an even integer, F_m is also an even integer. Thus $3 | m$ by (3.10) and therefore $m = 3a$ for some integer a. On the other hand, since F_m is an even integer, $4 | F_n$ and thus $6 | n$ by Theorem 3.5. Since $n = mk = 3ak,$ one gets $6 | 3ak,$ i.e., $2 | ak.$ Moreover, since k is an odd integer, it is seen that $2 | a.$ This implies that $2 | m,$ which is impossible since m is an odd integer. Assume that $3 | k.$ Then $k = 3s$ for some odd integer s. Therefore $n = mk = 3ms.$ Since ms is an odd integer, one obtains

$$F_n = F_{3ms} = F_{ms}(5F_{ms}^2 - 3) = 2F_m x^2$$

by (3.8). This shows that $(F_{ms}/F_m)(5F_{ms}^2 - 3) = 2x^2.$ It can be easily seen that if $d = (F_{ms} / F_m, 5F_{ms}^2 - 3)$, then $d = 1$ or $d = 3$. Assume that $d = 3.$ Then $3 | F_{ms},$ and thus $4 \mid ms$ by Theorem 3.5. But this is not possible while ms is an odd integer. So $d = 1.$ Then it follows that

$$\frac{F_{ms}}{F_m} = u^2,\ 5F_{ms}^2 - 3 = 2v^2 \tag{3.22}$$

or

$$\frac{F_{ms}}{F_m} = 2u^2,\ 5F_{ms}^2 - 3 = v^2 \tag{3.23}$$

for some integers u and v. Assume that (3.22) is satisfied. Then $ms = m,$ i.e., $s = 1$ by Theorem 3.1.1. Therefore $5F_m^2 - 3 = 2v^2$ and this shows that $2v^2 = 5F_m^2 - 3 = L_m^2 + 1 = L_{2m} - 1$ by (3.5) and (3.9). This implies that $L_{2m} = 2v^2 + 1.$ Since $L_{2m} = 2v^2 + 1,$ we get $3 \nmid m$. Thus $m = 6q \mp 1 = 3.2^{r+1} b \mp 1,$ where $q = 2^r b$ for some odd integer b with $r \geq 0.$ This shows that

$$L_{2m} = L_{2.2^{r+1}3b\mp 2} \equiv -L_{\mp 2}(mod L_{2^{r+1}})$$

and therefore

$$2v^2 + 1 \equiv -3(mod L_{2^{r+1}}),$$

i.e.,

$$2v^2 \equiv -4(mod L_{2^{r+1}})$$

by (3.3). On the other hand,

$$v^2 \equiv -2(mod L_{2^{r+1}})$$

since $L_{2^{r+1}}$ is an odd integer. This shows that $\left(\frac{-2}{p}\right) = 1$ for every prime divisor of $L_{2^{r+1}}$. Then it follows that

$$p \equiv 1,3(mod 8)$$

by (3.13) and therefore

$$L_{2^{r+1}} \equiv 1,3(mod 8).$$

This shows that $r = 0$ by Lemma 3.3. Consequently, q is an odd integer. Therefore it can be easily seen that $m = 12c + 5$ or $m = 12c + 7$ for some integer $c.$ Thus one arrives

$$L_m \equiv 3(mod 8)$$

or

$$L_m \equiv 5(mod 8)$$

by (3.2). On the other hand,

$$2v^2 = L_m^2 + 1$$

implies that

$$2v^2 \equiv 1(mod L_m),$$

and so

$$(2v)^2 \equiv 2(mod L_m).$$

Therefore $\left(\frac{2}{p}\right) = 1$ for every prime divisor of L_m. Then it follows that

$$p \equiv \mp 1(mod 8)$$

by (3.12) and hence

$$L_m \equiv \mp 1(mod 8).$$

But this contradicts the fact that $L_m \equiv 3,5(mod 8)$. Assume that (3.23) is satisfied. Then $v^2 = 5F_{ms}^2 - 3 = L_{ms}^2 + 1$ by (3.9). This implies that $L_{ms} = 0$, which is not possible. This completes the proof.

Theorem 3.1.5. If $L_n = 6L_m x^2$ and $m \geq 1$, then $m = 2$, $x^2 = 1$, and $n = 6$.

Proof: Assume that $L_n = 6L_m x^2$ for some integer x. Then $3 | L_n$ and therefore $n = 2k_0$ for some odd integer k_0 by Theorem 3.4. Moreover, since $2 | L_n$, one gets $3 | n$ by (3.10). This shows that $3 | k_0$ and therefore $k_0 = 3t$ for some odd integer t. Thus $n = 6t = 6(2u+1) = 12u + 6$. Hence,

$$L_n = L_{12u+6} \equiv L_6 (mod 8),$$

or

$$L_n \equiv 2(mod 8)$$

by (3.2). Since $8 \nmid L_n$, it can be seen that x is an odd integer. So

$$x^2 \equiv 1(mod 8),$$

which implies that

$$6L_m x^2 \equiv 6L_m (mod 8).$$

This shows that

$$6L_m \equiv 2(mod 8),$$

which implies that $m \neq 1$. Now assume that $m > 2$. Since $L_m \mid L_n$, there exists an odd integer k such that $n = mk$ by Theorem 3.4. On the other hand, since $2 \mid n$, it is seen that $2 \mid m$. Therefore $m = 2r$ for some odd integer r. If $r = 6q + 3$, then $m = 2r = 12q + 6$ and therefore

$$L_m \equiv L_6 (mod 8)$$

by (3.2). That is,

$$L_m \equiv 2(mod 8),$$

which is not possible since

$$6L_m \equiv 2(mod 8).$$

Therefore $3 \nmid r$. Since $n = mk$, $m = 2r$, and $3 \nmid r$, it follows that $3 \mid k$ and thus $k = 3s$ for some odd integer s. Then

$$L_n = L_{mk} = L_{3ms} = L_{ms}(L_{ms}^2 - 3) = 6L_m x^2$$

by (3.6). It can be seen that $(L_{ms}, L_{ms}^2 - 3) = 3$. Thus $\left(L_{ms}, \dfrac{L_{ms}^2 - 3}{3} \right) = 1$. Then

$$\frac{L_{ms}}{L_m} \left(\frac{L_{ms}^2 - 3}{3} \right) = 2x^2 ,$$

which shows that

$$\frac{L_{ms}}{L_m} = 2u^2 \text{ and } \frac{L_{ms}^2 - 3}{3} = v^2 \tag{3.24}$$

or

$$\frac{L_{ms}}{L_m} = u^2 \text{ and } \frac{L_{ms}^2 - 3}{3} = 2v^2. \tag{3.25}$$

for some integers u and v. Assume that (3.24) is satisfied. Then $3\left(\dfrac{L_{ms}}{3} \right)^2 - 1 = v^2$ and therefore

$$v^2 \equiv -1(mod 3),$$

which is a contradiction. Now assume that (3.25) is satisfied. Then $L_{ms} = L_m u^2$, which implies that $ms = m$ by Theorem 3.1.2. That is, $s = 1$. Thus $L_m^2 - 3 = 6v^2$. Since $L_m^2 = L_{2m} + 2$ by (3.5), it is seen that $L_{2m} - 1 = 6v^2$. Moreover, since $m = 2r$, it follows that $L_{4r} - 1 = 6v^2$. On the other hand, since $4r$ can be written as $4r = 4(4u \mp 1) = 16u \mp 4 = 2.2^b.a \mp 4$ for some odd integer a with $b \geq 3$, it follows that

$$L_{4r} = L_{2.2^b a \mp 4} \equiv -L_{\mp 4}(mod L_{2^b})$$

by (3.3) and therefore

$$1 + 6v^2 \equiv -7(mod L_{2^b}).$$

Then

$$6v^2 \equiv -8(mod L_{2^b}),$$

or

$$3v^2 \equiv -4(mod L_{2^b}),$$

or

$$(3v)^2 \equiv -12(mod L_{2^b}),$$

which shows that $\left(\frac{-12}{p}\right) = 1$ for every prime divisor of L_{2^b}. Then it follows that

$$p \equiv 1(mod 3)$$

and therefore

$$L_{2^b} \equiv 1(mod 3).$$

But this contradicts Lemma 3.2. Therefore $m = 2$. This completes the proof.

In [22], it is shown that $L_n = 3L_m x^2$ has no solutions if $m > 1$. Now a similar result is given for Fibonacci numbers.

Theorem 3.1.6. Let $m \geq 3$ be an integer and $F_n = 3F_m x^2$. Then $m = 4$, $x^2 = 16$, and $n = 12$.

Proof: Assume that $m \geq 3$ and $F_n = 3F_m x^2$ for some integer x. Then $F_m \mid F_n$ and therefore $n = mk$ for some integer k by Theorem 3.5.

Firstly, assume that k is an even integer. Then $k = 2s$ for some $s \in \mathbf{N}$. Therefore $n = mk = 2ms$. Thus

$$F_n = F_{2ms} = F_{ms} L_{ms} = 3F_m x^2$$

by (3.7). This shows that

$$(F_{ms}/F_m) L_{ms} = 3x^2.$$

It can be easily seen that if $(F_{ms}/F_m, L_{ms}) = d$, then $d = 1$ or $d = 2$ by (3.11). Let $d = 1$. Then

$$\frac{F_{ms}}{F_m} = u^2,\ L_{ms} = 3v^2 \tag{3.26}$$

or

$$\frac{F_{ms}}{F_m} = 3u^2,\ L_{ms} = v^2 \tag{3.27}$$

for some integers u and v. Let (3.26) be satisfied. Then $L_{ms} = 3v^2 = L_2 x^2$ and therefore $ms = 2$ by Theorem 3.1.2. But this is not possible since $m \geq 3$. Let (3.27) be satisfied. Then $ms = 3$ by Theorem 3.1. Thus $m = 3$ and $s = 1$. Then $3u^2 = \left(F_3/F_3\right) = 1$, which is not possible. Let $d = 2$. Then

$$\frac{F_{ms}}{F_m} = 2u^2,\ L_{ms} = 6v^2 \tag{3.28}$$

or

$$\frac{F_{ms}}{F_m} = 6u^2,\ L_{ms} = 2v^2 \tag{3.29}$$

for some integers u and v. The identity (3.28) is not possible by Theorem 3.1.3. Let (3.29) be satisfied. Then $ms = 6$ by Theorem 3.1. If $m = 6$ and $s = 1$, then this is impossible since $F_m = F_{ms} = 6F_m u^2$. If $m = 3$ and $s = 2$, then $6u^2 = F_6/F_3 = 4$, which is impossible.

Now suppose that k is an odd integer. Since $3 \mid F_n$, we get $4 \mid n$ by Theorem 3.5. Moreover, since $n = mk$ and k is odd, one gets $4 \mid m$. Let x be an even integer.

Then $4 \mid F_n$. Thus $L_3 \mid F_n$ and $3 \mid n$ by Theorem 3.6. Therefore, $12 \mid n$ since $4 \mid n$ and $3 \mid n$, i.e., $n = 12t$ for some $t \in \mathbf{N}$. On the other hand, $m = 4r$ for some $r \in \mathbf{N}$ since $4 \mid m$. Therefore $12t = n = mk = 4rk$. It follows that $3t = rk$. Thus

$$F_n = F_{12t} = F_{6t} L_{6t} = 3F_{2r} L_{2r} x^2$$

by (3.7). Since $(6t/2r) = k$ and k is odd, one can write

$$\frac{F_{6t}}{F_{2r}} . \frac{L_{6t}}{L_{2r}} = 3x^2 .$$

Assume that $3 \mid r$. Then, it can be seen that $\left(\dfrac{F_{6t}}{F_{2r}}, \dfrac{L_{6t}}{L_{2r}} \right) = 1$ by (3.11). Therefore

$$\frac{F_{6t}}{F_{2r}} = u^2 ,\ \frac{L_{6t}}{L_{2r}} = 3v^2 \qquad (3.30)$$

or

$$\frac{F_{6t}}{F_{2r}} = 3u^2 ,\ \frac{L_{6t}}{L_{2r}} = v^2 \qquad (3.31)$$

for some integers u and v. A similar argument shows that (3.30) and (3.31) are impossible. Now assume that $3 \nmid r$. Then since $3t = rk$, it follows that $3 \mid k$. Thus $k = 3s$ for some $s \in \mathbf{N}$. Then $3t = rk = 3rs$ and therefore $t = rs$. Also since $3 \nmid r$, it can be seen that $\left(\dfrac{F_{6t}}{F_{2r}}, \dfrac{L_{6t}}{L_{2r}} \right) = 2$ by (3.11). Therefore

$$\frac{F_{6t}}{F_{2r}} = 2u^2 ,\ \frac{L_{6t}}{L_{2r}} = 6v^2 \qquad (3.32)$$

or

$$\frac{F_{6t}}{F_{2r}} = 6u^2 ,\ \frac{L_{6t}}{L_{2r}} = 2v^2 \qquad (3.33)$$

for some integers u and v. Assume that (3.32) is satisfied. Then $2r = 2$ by Theorem 3.1.5. This shows that $r = 1$ and $t = s$. Thus $L_{6t} = 6L_2 v^2 = L_6 v^2$, which implies that $6t = 6$, i.e., $t = 1$ by Theorem 3.1.2. Therefore $k = 3s = 3t = 3$ and $m = 4r = 4$. Therefore $n = 12$ and $x^2 = 16$. Now assume that (3.33) is satisfied. Then it follows that

$$L_{6t} = 2L_{2r} v^2 ,$$

which is impossible by Theorem 3.1.3. Now, let x be an odd integer. Then

$$F_n \equiv 3F_m (mod 8).$$

Since $4 | m$, it follows that $m = 12q$ or $m = 12q \mp 4$ for some integer q. If $m = 12q \mp 4$, then

$$F_m \equiv F_{12q \mp 4} \equiv F_{\mp 4} \equiv \mp 3 (mod 8)$$

by (3.1). Therefore

$$F_n \equiv \mp 1 (mod 8),$$

which is impossible since $4 | n$. Because if $4 | n$, then $n = 12r \mp 4$ or $n = 12r$ for some integer r, and therefore $F_n \equiv \mp 3, 0 (mod 8)$ by (3.1). If $m = 12q$, then $n = mk = 12qk$. This shows that $6qk/6q$ is an odd integer. Then, from the equality

$$F_n = F_{12qk} = F_{6qk} L_{6qk} = 3F_m x^2 = 3F_{6q} L_{6q} x^2,$$

it follows that

$$\frac{F_{6qk}}{F_{6q}} . \frac{L_{6qk}}{L_{6q}} = 3x^2.$$

Since $\left(\frac{F_{6qk}}{F_{6q}}, \frac{L_{6qk}}{L_{6q}} \right) = 1$, one has

$$\frac{F_{6qk}}{F_{6q}} = u^2, \ \frac{L_{6qk}}{L_{6q}} = 3v^2 \qquad (3.34)$$

or

$$\frac{F_{6qk}}{F_{6q}} = 3u^2, \ \frac{L_{6qk}}{L_{6q}} = v^2 \qquad (3.35)$$

for some integers u and v. Similarly, it can be seen that the identities (3.34) and (3.35) are impossible. This completes the proof.

Finally, we can give the following theorem without proof since the proof is similar to that of Theorem 3.1.6.

Theorem 3.1.7. There is no integer x such that $F_n = 6F_m x^2$.

CHAPTER 4. THE SQUARE TERMS IN GENERALIZED FIBONACCI AND LUCAS SEQUENCES

In this chapter, we solve the generalization of the equations $L_n = 2L_m x^2$, $L_n = 3L_m x^2$ and $L_n = 6L_m x^2$. Also, using congruences related to generalized Fibonacci and Lucas numbers given in the second chapter, some equations including generalized Fibonacci and Lucas numbers are solved under some assumptions.

4.1. Some Fundamental Theorems and Identities

In this subsection, some theorems, lemmas, and some identities about generalized Fibonacci and Lucas numbers which will be used later are given.

Since the proof of the following lemma can be proved by induction, the proof is omitted.

Lemma 4.1.1. Let $Q \equiv 1 (\bmod 3)$. If $3 \nmid P$, then

$$V_{2^r} \equiv \begin{cases} 2(mod 3) & \text{if } r \geq 3, \\ 1(mod 3) & \text{if } r = 2, \\ 0(mod 3) & \text{if } r = 1, \end{cases}$$

and if $3 \mid P$, then $V_{2^r} \equiv 2(mod 3)$ for all $r \geq 1$.

The following two theorems can be given from Corollary 2.1.1 and Corollary 2.1.2.

Theorem 4.1.1. Let $Q = 1$, $n \in \mathbf{N} \cup \{0\}$, $m, r \in \mathbf{Z}$, and m be nonzero integer. Then

$$U_{2mn+r} \equiv (-1)^{mn} U_r (mod U_m) \tag{4.1}$$

and

$$V_{2mn+r} \equiv (-1)^{mn} V_r \left(mod U_m\right). \tag{4.2}$$

Theorem 4.1.2. Let $Q = 1$, $n \in \mathbf{N} \cup \{0\}$, and $m, r \in \mathbf{Z}$. Then

$$U_{2mn+r} \equiv (-1)^{(m+1)n} U_r \left(mod V_m\right) \tag{4.3}$$

and

$$V_{2mn+r} \equiv (-1)^{(m+1)n} V_r \left(mod V_m\right). \tag{4.4}$$

When P and Q are odd, using (2.18) and (2.19), one gets $8 \mid U_6$ and therefore

$$U_{12q+r} \equiv U_r \left(mod 8\right) \tag{4.5}$$

and

$$V_{12q+r} \equiv V_r \left(mod 8\right) \tag{4.6}$$

for nonnegative integers q and r. It can be seen from (4.6) that if P and Q are odd and $Q \not\equiv 5 (\mathrm{mod}\, 8)$, then

$$8 \nmid V_n, \tag{4.7}$$

and if P and Q are odd and $Q \not\equiv 1,5 (\mathrm{mod}\, 8)$, then

$$4 \nmid V_n \tag{4.8}$$

for every natural number n.

The proof of the following lemma is given in [42]. Moreover, the lemma can be proved by using Corollary 2.1.1 and Corollary 2.1.2.

Lemma 4.1.2. Let $n \geq 1$. Then

$$(a)\quad 3 \mid V_n \Leftrightarrow \begin{cases} n \equiv 1 (mod\, 2) & \text{if } 3 \mid P, \\ n \equiv 2 (mod\, 4)\ and\ Q \equiv 1 (\mathrm{mod}\, 3) & \text{if } 3 \nmid P, \end{cases}$$

$$(b)\quad 3 \mid U_n \Leftrightarrow \begin{cases} n \equiv 0 (mod\, 2) & \text{if } 3 \mid P, \\ 4 \mid n\ and\ Q \equiv 1 (\mathrm{mod}\, 3)\ or\ 3 \mid n\ and\ Q \equiv 2 (\mathrm{mod}\, 3) & \text{if } 3 \nmid P. \end{cases}$$

The following lemma can be found in [42].

Lemma 4.1.3. Let P, Q, and m be odd positive integers and $r \geq 1$.

(a) If $3 \nmid m$, then $V_{2^r m} \equiv \begin{cases} 3(mod\,8), & \text{if } r = 1 \text{ and } Q \equiv 1(mod\,4), \\ 7(mod\,8), & \text{otherwise,} \end{cases}$

(b) If $3 \mid m$, then $V_{2^r m} \equiv 2(mod\,8)$.

By the above lemma, when P and Q are odd, it is seen that

$$\left(\frac{-1}{V_{2^r}}\right) = -1 \tag{4.9}$$

for $r \geq 1$.

The following lemma can be proved by induction.

Lemma 4.1.4. If n is a positive even integer, then $V_n \equiv 2Q^{\frac{n}{2}}(mod\,P^2)$ and if n is a positive odd integer, then $V_n \equiv nPQ^{\frac{n-1}{2}}(mod\,P^2)$.

Now, the following identities concerning generalized Fibonacci and Lucas numbers can be given.

$$U_{2n} = U_n V_n, \tag{4.10}$$

$$V_{2n} = V_n^2 - 2(-Q)^n, \tag{4.11}$$

$$U_{3n} = U_n((P^2 + 4Q)U_n^2 + 3(-Q)^n) = U_n(V_n^2 - (-Q)^n), \tag{4.12}$$

$$V_{3n} = V_n(V_n^2 - 3(-Q)^n), \tag{4.13}$$

$$\text{If } P \text{ is odd and } n \geq 1, \text{ then } 2 \mid V_n \Leftrightarrow 2 \mid U_n \Leftrightarrow 3 \mid n, \tag{4.14}$$

$$V_n^2 - (P^2 + 4Q)U_n^2 = 4(-Q)^n, \tag{4.15}$$

$$\text{If } (P,Q) = 1 \text{ and } n \geq 1, \text{ then } (U_n, Q) = (V_n, Q) = 1. \tag{4.16}$$

Let $m = 2^a k$, $n = 2^b l$, k and l odd, $a, b \geq 0$, and $d = (m,n)$. Then

$$(U_n, U_m) = U_d, \tag{4.17}$$

$$(U_m, V_n) = \begin{cases} V_d & \text{if } a > b, \\ 1 \text{ or } 2 & \text{if } a \leq b, \end{cases} \tag{4.18}$$

$$(V_m, V_n) = \begin{cases} V_d & \text{if } a = b, \\ 2 & \text{if } a \neq b \text{ and } P \text{ is even}, \\ 1 \text{ or } 2 & \text{if } a \neq b \text{ and } P \text{ is odd}. \end{cases} \tag{4.19}$$

$$\text{If } U_m \neq 1, \text{ then } U_m \mid U_n \Leftrightarrow m \mid n, \tag{4.20}$$

$$\text{If } V_m \neq 1, \text{ then } V_m \mid V_n \Leftrightarrow m \mid n \text{ and } \frac{n}{m} \text{ is odd}, \tag{4.21}$$

$$\text{If } V_m \neq 1, \text{ then } V_m \mid U_n \Leftrightarrow m \mid n \text{ and } \frac{n}{m} \text{ is even}, \tag{4.22}$$

$$\text{If } P \text{ is odd, } Q = 1,\ 2 \mid m, \text{ and } 3 \nmid m, \text{ then} \left(\frac{V_3}{V_m}\right) = \left(\frac{-2}{P}\right), \tag{4.23}$$

If P and Q are odd, then

$$\left(\frac{U_3}{V_{2^r}}\right) = 1 \Leftrightarrow r > 1, \text{ or } r = 1 \text{ and } Q \equiv 3 (mod\, 4), \tag{4.24}$$

$$\text{If } r \geq 3 \text{ and } Q = 1, \text{ then } V_{2^r} \equiv 2 (mod\, V_2). \tag{4.25}$$

Moreover,

$$\text{if } P > 1, \text{ then } V_m \neq 1 \text{ for all } m \in \mathbf{N}$$

and also when P is even, it can be easily seen that

$$U_n \text{ is odd} \Leftrightarrow n \text{ is odd},$$

$$U_n \text{ is even} \Leftrightarrow n \text{ is even},$$

$$V_n \text{ is even for all } n \in \mathbf{N}.$$

Identities between (4.10)-(4.16) and (4.17)-(4.22) can be found in [41], [42], [47] and [29], [40], [42], respectively. Identities (4.23) and (4.24) are given in [2], [9] and [42], respectively. The proofs of the others are easy and will be omitted.

From now on, we will assume that n is a positive integer.

4.2. Generalized Lucas Numbers of The Form cx^2

In this subsection, it is assumed that $Q = 1$ and $P > 1$. Firstly, when $k \mid P$ and P is odd, indices n such that $V_n = kx^2$ are determined. Then, when P is odd, it is shown that there is no solutions of the equation $V_n = 3x^2$ for $n > 2$. Moreover, it is proved that the equation $V_n = 6x^2$ has no solutions when P is odd. Finally, the equations $V_n = 3V_m x^2$ and $V_n = 6V_m x^2$ are considered. It is shown that the equation $V_n = 3V_m x^2$ has solutions when $n = 3$, $m = 1$, and P is odd. Also, it is shown that the equation $V_n = 6V_m x^2$ has solutions only when $n = 6$. Also the equations $V_n = 3x^2$ and $V_n = 3V_m x^2$ are considered under some assumptions when P is even.

In [11], Cohn solved the equations $V_n = V_m x^2$ and $V_n = 2V_m x^2$ when P is odd. Now, the following two theorems which can be obtained from Theorem 11 and Theorem 12 in [11] are given.

Theorem 4.2.1. Let P be an odd integer, $m \geq 1$ be an integer, and $V_n = V_m x^2$ for some integer x. Then $n = m$.

Theorem 4.2.2. If P is odd, then there is no integer x such that $V_n = 2V_m x^2$ for $m \geq 1$.

In the following theorem, it is shown that the equation $V_n = 2V_m x^2$ has no solutions when P is even.

Theorem 4.2.3. If P is even, then there is no integer x such that $V_n = 2V_m x^2$ for $m \geq 1$.

Proof: Assume that P is even and $V_n = 2V_m x^2$. Then V_m is even, which implies that $4 | V_n$. Therefore, it is seen that n is odd by Lemma 4.1.4. Moreover, since $V_m | V_n$, there exists an odd integer t such that $n = mt$ by (4.21). Thus, m is odd and therefore $V_n \equiv nP (mod P^2)$ and $V_m \equiv mP (mod P^2)$ by Lemma 4.1.4. It follows that $nP \equiv 2mPx^2 (mod P^2)$, i.e., $n \equiv 2mx^2 (mod P)$. This is a contradiction since n is odd. This completes the proof.

The proof of the following lemma can be seen from identity (4.23).

Lemma 4.2.1. If P is odd and $r \geq 1$, then $\left(\frac{P^2+3}{V_{2^r}}\right)=1.$

Theorem 4.2.4. Let k be square-free positive divisor of P and P be an odd integer. If the equation $V_n = kx^2$ has a solution for some integer x, then $n=1$ or $n=3$.

Proof: Assume that $k \mid P$ and $V_n = kx^2$. Then, it is seen that n is odd by Lemma 4.1.4. Let $n>3$. Then $n=4q+1$ or $n=4q+3$ for some $q>0$. Also, $P=kM$ for some positive integer M since $k \mid P$. For the remaining part of the proof, two cases can be considered.

Case 1: Assume that $n=4q+1$. For some odd integer z, $n=4q+1=2(2^r z)+1$, where $r \geq 1$. Thus one gets

$$V_n \equiv -P(modV_{2^r}),$$

i.e.,

$$kx^2 \equiv -P(modV_{2^r})$$

by (4.4). This shows that $J=\left(\frac{-kP}{V_{2^r}}\right)=1$. Since $P=kM$, it follows that $J=\left(\frac{-kP}{V_{2^r}}\right)=\left(\frac{-k^2M}{V_{2^r}}\right)=\left(\frac{-M}{V_{2^r}}\right)$. Since $V_{2^r} \equiv 2(modP)$ by Lemma 4.1.4, it is seen that $V_{2^r} \equiv 2(modM)$ and thus $(M,V_{2^r})=1$. Also $\left(\frac{-1}{V_{2^r}}\right)=-1$ by (4.9). Assume that $M \equiv 1,3(mod8)$. Hence

$$J=\left(\frac{-M}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)=(-1)\left(\frac{V_{2^r}}{M}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{M-1}{2}\right)}$$

$$= (-1)\left(\frac{2}{M}\right)(-1)^{\left(\frac{M-1}{2}\right)} = (-1)(-1)^{\left(\frac{M^2-1}{8}\right)}(-1)^{\left(\frac{M-1}{2}\right)} = -1.$$

This contradicts the fact that $J = 1$. Assume that $M \equiv 5,7(mod 8)$. Since $n = 4q+1 = 4(q+1)-3 = 2(2^r z)-3$ for some odd integer z with $r \geq 1$, it follows that

$$V_n \equiv V_3(modV_{2^r}),$$

i.e.,

$$kx^2 \equiv V_3(modV_{2^r})$$

by (4.4). This shows that

$$1 = J = \left(\frac{kV_3}{V_{2^r}}\right) = \left(\frac{k^2M(P^2+3)}{V_{2^r}}\right) = \left(\frac{M(P^2+3)}{V_{2^r}}\right).$$

Since $M \equiv 5,7(mod 8)$, $V_{2^r} \equiv 3,7(mod 8)$ by Lemma 4.1.3, and $V_{2^r} \equiv 2(modM)$ by Lemma 4.1.4, it follows that

$$\left(\frac{M}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{M}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{M-1}{2}\right)}$$

$$= \left(\frac{2}{M}\right)(-1)^{\left(\frac{M-1}{2}\right)} = (-1)^{\left(\frac{M^2-1}{8}\right)}(-1)^{\left(\frac{M-1}{2}\right)} = -1.$$

Thus, using Lemma 4.2.1, one obtains

$$J = \left(\frac{kV_3}{V_{2^r}}\right) = \left(\frac{k^2M(P^2+3)}{V_{2^r}}\right) = \left(\frac{M(P^2+3)}{V_{2^r}}\right) = \left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2+3}{V_{2^r}}\right) = -1,$$

which is a contradiction.

Case 2: Assume that $n = 4q+3$. For some odd integer z, one can write $n = 2(2^r z)+3$, where $r \geq 1$. Thus

$$V_n \equiv -V_3(modV_{2^r}),$$

i.e.,

$$kx^2 \equiv -V_3(modV_{2^r})$$

by (4.4). This shows that

$$1 = J = \left(\frac{-kV_3}{V_{2^r}}\right) = \left(\frac{-k^2M(P^2+3)}{V_{2^r}}\right) = \left(\frac{-M(P^2+3)}{V_{2^r}}\right).$$

Assume that $M \equiv 1,3(mod 8)$. Then, it can be shown that

$$\left(\frac{M}{V_{2^r}}\right) = 1.$$

Thus, the identity (4.9) and Lemma 4.2.1 imply that

$$J = \left(\frac{kV_3}{V_{2^r}}\right) = \left(\frac{-M(P^2+3)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2+3}{V_{2^r}}\right) = -1,$$

which is a contradiction. Assume that $M \equiv 5,7(mod 8)$. Since $n = 4q+3 = 4(q+1)-1 = 2(2^r z)-1$ for some odd integer z with $r \geq 1$, one gets

$$V_n \equiv P(mod V_{2^r}),$$

i.e.,

$$kx^2 \equiv P(mod V_{2^r})$$

by (4.4). This shows that $1 = J = \left(\frac{kP}{V_{2^r}}\right) = \left(\frac{k^2M}{V_{2^r}}\right) = \left(\frac{M}{V_{2^r}}\right)$. On the other hand, since $M \equiv 5,7(mod 8)$, it is seen that

$$\left(\frac{M}{V_{2^r}}\right) = (-1)^{\left(\frac{M^2-1}{8}\right)}(-1)^{\left(\frac{M-1}{2}\right)} = -1,$$

which contradicts the fact that $J = 1$. This completes the proof.

In [42], Ribenboim and McDaniel have solved the equation $U_n = 3x^2$ for all relatively prime odd integers P and Q. Also, for $P \equiv 1,3(mod 8)$, $Q \equiv 3(mod 4)$ and $(P,Q) = 1$, the solutions of the equation $V_n = 3x^2$ is given by Ribenboim and McDaniel in [42]. In [39], the same authors have shown that $V_n \neq 3x^2$ and $V_n \neq 6x^2$ for all odd relatively prime values of P and Q with $Q \equiv 3(mod 4)$ and $n \equiv \pm 3(mod 8)$. Besides, in [3], Antoniadis has solved the equation $V_n = 3x^2$ for $3 \nmid P$ with P odd and $Q = 1$. Now the proof of the following theorem is given in a different way for the sake of completeness.

Theorem 4.2.5. If P is odd, then the equation $V_n = 3x^2$ has the solutions for $n = 1$ or $n = 2$ and, if P is even and $3 \nmid P$, then there is no integer x such that $V_n = 3x^2$.

Proof: Assume that $3 \nmid P$. Since $3 | V_n$, it follows that $n \equiv 2(mod 4)$ by Lemma 4.1.2. If P is even, then $3x^2 = V_n \equiv 2(mod 4)$ by Lemma 4.1.4 since n is even. But this is not possible. Now, assume that P is odd. If $n = 2$, then $V_n = V_2 = P^2 + 2 = 3x^2$ or $P^2 - 3x^2 = -2$. Since $(u_1, v_1) = (1,1)$ is the fundamental solution of the equation $u^2 - 3v^2 = -2$, all positive integer solutions of the equation $u^2 - 3v^2 = -2$ are given by

$$(u, v) = \left(5U_m(4,-1) - U_{m-1}(4,-1), 3U_m(4,-1) - U_{m-1}(4,-1)\right)$$

with $m \geq 0$. Therefore, when $n = 2$, the equation $V_n = 3x^2$ has a solution for $P = 5U_m(4,-1) - U_{m-1}(4,-1)$. If $n = 6$, then $3x^2 = V_6 = V_3^2 + 2$ by (4.11). Since P is odd, V_3 is even by (4.14) and therefore

$$3x^2 = V_3^2 + 2 \equiv 2(mod 4),$$

which is not possible. Then it is clear that $n = 16c \pm 2$ or $n = 16c \pm 6$ for some positive integer c. Let $n = 16c \pm 6$. Thus

$$V_n \equiv V_6 (mod V_4),$$

i.e.,

$$3x^2 \equiv V_6 (mod V_4) \tag{4.26}$$

by (4.4). Moreover, it can be easily shown that $V_6 \equiv -V_2 (mod V_4)$. Hence $3x^2 \equiv -V_2 (mod V_4)$ from (4.26). Then $J = \left(\frac{-3V_2}{V_4}\right) = 1$. On the other hand, since $\left(\frac{-1}{V_{2^r}}\right) = -1$ for $r \geq 1$ by (4.9), $V_4 \equiv 1(mod 3)$ by Lemma 4.1.1, and $V_4 \equiv -2(mod V_2)$ by (4.11), it is seen that

$$\left(\frac{-1}{V_4}\right) = (-1)^{\frac{V_4 - 1}{2}} = -1,$$

$$\left(\frac{3}{V_4}\right) = \left(\frac{V_4}{3}\right)(-1)^{\frac{V_4-1}{2}} = -1,$$

and

$$\left(\frac{V_2}{V_4}\right) = \left(\frac{V_4}{V_2}\right)(-1)^{\left(\frac{V_4-1}{2}\right)\left(\frac{V_2-1}{2}\right)} = \left(\frac{-2}{V_2}\right)(-1) = \left(\frac{-1}{V_2}\right)\left(\frac{2}{V_2}\right)(-1) = -1.$$

These imply that

$$J = \left(\frac{-3V_2}{V_4}\right) = \left(\frac{-1}{V_4}\right)\left(\frac{3}{V_4}\right)\left(\frac{V_2}{V_4}\right) = (-1)(-1)(-1) = -1,$$

which contradicts the fact that $J = 1$. Now let $n = 16c \pm 2$. Since n can be written as $n = 2(2^r z) \pm 2$ for some odd z with $r \geq 3$, it is seen that

$$V_n \equiv -V_2 (mod V_{2^r}),$$

i.e.,

$$3x^2 \equiv -V_2 (mod V_{2^r})$$

by (4.4), which shows that $J = \left(\frac{-3V_2}{V_{2^r}}\right) = 1$. On the other hand, $V_{2^r} \equiv 2(mod V_2)$ for $r \geq 3$ by (4.25) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (4.9). Thus

$$\left(\frac{V_2}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{V_2}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{V_2-1}{2}\right)} = \left(\frac{2}{V_2}\right)(-1) = 1.$$

Moreover, $V_{2^r} \equiv 2(mod 3)$ by Lemma 4.1.1. Then

$$\left(\frac{3}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{3}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{3-1}{2}\right)} = \left(\frac{2}{3}\right)(-1) = 1$$

and hence

$$J = \left(\frac{-3V_2}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{V_2}{V_{2^r}}\right) = -1,$$

which is a contradiction.

Now assume that $3 \mid P$. Then $n=1$ or $n=3$ by Theorem 4.2.4. If $n=1$, then $V_1 = P = 3x^2$. It is obvious that this is a solution. If $n=3$, then $V_3 = P(P^2+3) = 3x^2$. Also, since $3 \mid P$, it is seen that $\left(P,(P^2+3)/3\right)=1$. Therefore $P=a^2$ and $P^2+3=3b^2$ for some positive integers a and b. Since $3 \mid P$, $P=3c$ for some positive integer c. Hence one obtains the Pell equation $b^2-3c^2=1$. It is well known that this equation has the solution $(b,c)=\left((V_m(4,-1))/2, U_m(4,-1)\right)$ for $m \geq 1$. It follows that $P=a^2=3c=3U_m$, i.e., $U_m=3x^2$. It is seen that the equation $U_m(4,-1)=3x^2$ has no solutions by Theorem 2 given in [34]. Therefore, the case for when $n=3$ is not possible. This completes the proof.

Theorem 4.2.6. If P is odd, then there is no integer x such that $V_n = 6x^2$.

Proof: Assume that $3 \nmid P$. If x is even, it follows that $8 \mid V_n$, which is impossible by (4.7). Therefore x is odd. Since $3 \mid V_n$ and $2 \mid V_n$, it is seen that $n/2$ is odd by Lemma 4.1.2 and $3 \mid n$ by (4.14), respectively. Then it follows that $n=12q+6$ for some positive integer q. Thus

$$V_n \equiv V_6 (mod 8)$$

by (4.6). That is,

$$6x^2 \equiv 2 (mod 8),$$

which is impossible since x is odd.

Assume that $3 \mid P$. Since $3 \mid V_n$, n is odd by Lemma 4.1.2. Also since $3 \mid n$, it follows that $n \equiv \pm 3 (mod 12)$. Thus

$$V_n \equiv V_{\pm 3} \equiv \pm 4P \equiv 4 (mod 8)$$

by (4.6). That is,

$$6x^2 \equiv 4 (mod 8),$$

which is impossible since x is odd. This completes the proof.

Theorem 4.2.7. If P is odd, $m \geq 1$ and $V_n = 3V_m x^2$, then $m=1$ and $n=3$.

Proof: Assume that $3\nmid P$. Since $V_m | V_n$ and $3|V_n$, it follows that $n = mt$ for some odd positive integer t by (4.21) and $n/2$ is odd by Lemma 4.1.2, respectively. Therefore m is even and $m/2$ is odd. Then $m = 12q \pm 2$ or $m = 12q + 6$ for some positive integer q. Thus

$$V_m \equiv V_2, V_6 \equiv 2,3(mod 8)$$

by (4.6). Similarly, it is seen that $V_n \equiv 2,3(mod 8)$. Also, since $3x^2 \equiv 0,3,4(mod 8)$, it follows that $V_n = 3V_m x^2 \equiv 0,1,4,6(mod 8)$, which contradicts the fact that $V_n \equiv 2,3(mod 8)$.

Now assume that $3|P$. Since $3|V_n$, it is seen that n is odd by Lemma 4.1.2. Therefore m is also odd. Now, two different cases can be considered.

Case 1: Assume that $3|t$. Then $t = 3s$ for some odd integer s. Thus $n = mt = 3ms$. By (4.13), one gets $V_n = V_{3ms} = V_{ms}(V_{ms}^2 + 3) = 3V_m x^2$. Since ms is odd and $3|P$, it follows that $(V_{ms}/V_m)((V_{ms}^2 + 3)/3) = x^2$. It can be seen that $(V_{ms}/V_m, (V_{ms}^2 + 3)/3) = 1$. Therefore $V_{ms} = V_m a^2$ and $V_{ms}^2 + 3 = 3b^2$ for some positive integers a and b. Then $ms = m$, i.e., $s = 1$ by Theorem 4.2.1. Thus $V_m^2 + 3 = 3b^2$. Using (4.11), one obtains $V_{2m} = 3b^2 - 1$. Assume that $m > 1$. Then $2m = 2(4q \pm 1) = 2(2^r z) \pm 2$ for some odd integer z with $r \geq 2$. Hence

$$V_{2m} \equiv -V_2 \equiv -(P^2 + 2)(mod V_{2^r})$$

by (4.4). That is,

$$3b^2 \equiv -(P^2 + 2 - 1) \equiv -U_3 (mod V_{2^r}),$$

which shows that $J = \left(\dfrac{-3U_3}{V_{2^r}}\right) = 1$. On the other hand, $\left(\dfrac{-1}{V_{2^r}}\right) = -1$ by (4.9). Moreover, since $3|P$, it follows that $V_{2^r} \equiv 2(mod 3)$ by Lemma 4.1.1 and therefore

$$\left(\frac{3}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{3}\right)(-1)^{\frac{V_{2^r}-1}{2}} = \left(\frac{2}{3}\right)(-1) = 1.$$

Using (4.24), one gets

$$J = \left(\frac{-3U_3}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{U_3}{V_{2^r}}\right) = -1,$$

which is a contradiction. Thus $m = 1$, i.e., $n = 3$. Then the equation $V_n = 3V_m x^2$ yields $P^2 + 3 = 3x^2$. Moreover $P = 3c$ for some positive integer c since $3 \mid P$. Hence the Pell equation $x^2 - 3c^2 = 1$ is obtained. It can be seen that this equation has the solutions $(x,c) = \left(\left(V_k(4,-1)\right)/2, U_k(4,-1)\right)$ with $k \geq 1$. Therefore $m = 1$ and $n = 3$ is a solution.

Case 2: Assume that $3 \nmid t$. It is obvious that $t > 1$ and so $t = 6q \pm 1$ for some positive integer q. Then $n = mt = 2(3mq) \pm m$. Hence

$$V_n \equiv \pm V_m (mod V_{3m}),$$

i.e.,

$$3V_m x^2 \equiv \pm V_m (mod V_{3m})$$

by (4.4). It follows that

$$3V_m x^2 \equiv \pm V_m (mod V_m (V_m^2 + 3))$$

by (4.13), which implies that

$$3x^2 \equiv \pm 1 (mod V_m^2 + 3).$$

Since $3 \mid P$ and m is odd, it is seen that $3 \mid V_m$ by Lemma 4.1.2. Therefore

$$3x^2 \equiv \pm 1 (mod 3),$$

which is not possible. This completes the proof.

Theorem 4.2.8. If P is even, $3 \nmid P$, and $m \geq 1$, then there is no integer x such that $V_n = 3V_m x^2$.

Proof: Assume that P is even, $3 \nmid P$, and $V_n = 3V_m x^2$. Since $V_m \mid V_n$ and $3 \mid V_n$, there exist two odd positive integers t and z such that $n = mt$ and $n = 2z$ by (4.21) and Lemma 4.1.2, respectively. Therefore $m = 2r$ for some odd positive integer r. It is obvious that $t > 1$. Then $t = 4q \pm 1$ for some positive integer q. Thus, since $n = mt = 4mq \pm m$, one gets

$$V_n \equiv \pm V_m (mod V_{2m}),$$

by (4.4), i.e.,

$$3V_m x^2 \equiv \pm V_m (mod V_{2m}).$$

It is seen that $(V_m, V_{2m}) = 2$ by (4.19). Hence

$$3x^2 \equiv \pm 1 (mod V_{2m}/2).$$

Since $2m/4 = 4r/4 = r$ is odd, it is seen that $V_4 | V_{2m}$ by (4.21) and hence

$$3x^2 \equiv \pm 1 (mod V_4/2),$$

which shows that $J = \left(\frac{\pm 3}{V_4/2}\right) = 1.$ On the other hand, since $V_4 = P^4 + 4P^2 + 2$ and P is even, it is seen that $V_4 \equiv 2 (mod 8)$ and therefore $\left(\frac{-1}{V_4/2}\right) = (-1)^{\left(\frac{V_4-2}{4}\right)} = 1.$ Moreover, using Lemma 4.1.1, it is seen that $V_4/2 \equiv 2 (mod 3).$ Thus,

$$\left(\frac{3}{V_4/2}\right) = \left(\frac{V_4/2}{3}\right)(-1)^{\frac{V_4-2}{4}} = \left(\frac{2}{3}\right) = -1$$

or $J = -1,$ which is a contradiction. This completes the proof.

Theorem 4.2.9. If P is even, $3 | P,$ $2 \| P$, and $m \geq 1,$ then there is no integer x such that $V_n = 3V_m x^2.$

Proof: Assume that P is even, $3 | P,$ $2 \| P$, and $V_n = 3V_m x^2.$ Since $V_m | V_n$ and $3 | V_n,$ there exists an odd positive integer t such that $n = mt$ by (4.21) and n is odd by Lemma 4.1.2, respectively. Therefore m is also odd. It is obvious that $t > 1.$ Assume that $3 | t.$ Then $t = 3s$ for some odd positive integer $s.$ Since $n = mt = 3ms,$ one gets

$$3V_m x^2 = V_n = V_{3ms} = V_{ms}(V_{ms}^2 + 3)$$

by (4.13). Since $3 | P$ and ms is odd, it follows that $3 | V_{ms}$ by Lemma 4.1.2. Thus

$$(V_{ms}/V_m)((V_{ms}^2 + 3)/3) = x^2.$$

It can be easily seen that $(V_{ms}/V_m, (V_{ms}^2 + 3)/3) = 1$, which shows that

$$V_{ms} = V_m u^2 \text{ and } V_{ms}^2 + 3 = 3v^2$$

for some positive integers u and v. So $3v^2 = V_{ms}^2 + 3 = V_{2ms} + 1$ by (4.11). Thus

$$3v^2 \equiv 1 (mod V_{2ms}).$$

Since $V_2 | V_{2ms}$, i.e., $(P^2 + 2) | V_{2ms}$ by (4.21), it follows that

$$3v^2 \equiv 1 (mod P^2 + 2),$$

which shows that

$$J = \left(\frac{3}{(P^2 + 2)/2} \right) = 1.$$

On the other hand, since $3 | P$, it is seen that $(P^2 + 2)/2 \equiv 1 (mod 3)$. Therefore

$$J = \left(\frac{3}{(P^2 + 2)/2} \right) = \left(\frac{(P^2 + 2)/2}{3} \right) (-1)^{\frac{P^2}{4}} = (-1)^{\frac{P^2}{4}}.$$

Since $2 \| P$, it follows that $J = -1$, which is a contradiction. Now assume that $3 \nmid t$. Then $t = 6q \pm 1$ for some positive integer q since $t > 1$. Thus $n = mt = 2(3mq) \pm m$, which leads to

$$V_n \equiv \pm V_m (mod V_{3m}),$$

or

$$3V_m x^2 \equiv \pm V_m (mod V_{3m})$$

by (4.4). It follows that

$$3V_m x^2 \equiv \pm V_m (mod V_m (V_m^2 + 3))$$

by (4.13), i.e.,

$$3x^2 \equiv \pm 1 (mod V_m^2 + 3).$$

Since $3 | P$ and m is odd, it is seen that $3 | V_m$ by Lemma 4.1.2. Therefore

$$3x^2 \equiv \pm 1 (mod 3),$$

which is not possible. This completes the proof.

The following lemma, which can be proved by induction, will be used in the sequel.

Lemma 4.2.2. If P is odd and $k \geq 2$, then $V_{2^k} \equiv -1 (mod V_4 + 1)$.

Theorem 4.2.10. If $V_n = 6V_m x^2$ and $m \geq 1,$ then $m = 2$ and $n = 6.$

Proof: Firstly, assume that P is odd. Let $3 \nmid P$. Then, since $3 | V_n$ and $2 | V_n,$ it follows that $n = 2z$ for some odd integer z by Lemma 4.1.2 and $3 | n$ by (4.14), respectively. This shows that $3 | z$ and therefore $z = 3a$ for some odd positive integer $a.$ Thus $n = 2z = 6a = 12q + 6$ for some positive integer $q.$ Hence

$$V_n \equiv V_6 \equiv 2(mod 8)$$

by (4.6). Clearly, x is odd. Therefore $6V_m \equiv 6V_m x^2 = V_n \equiv 2(mod 8).$ Besides, since $V_m | V_n,$ there exists an odd positive integer t such that $n = mt$ by (4.21). Since $n = 2z$ with odd $z,$ it follows that $m = 2r$ for some odd positive integer $r.$ Assume that $3 | r.$ Then $V_m \equiv 2(mod 8)$ by Lemma 4.1.3 and therefore $6V_m \equiv 4(mod 8),$ which contradicts the fact that $6V_m \equiv 2(mod 8).$ Assume that $3 \nmid r$. Then, since $3 | n$ and $n = mt,$ it is seen that $3 | t,$ i.e., $t = 3s$ for some odd positive integer s. Thus $n = mt = 3ms$. Therefore

$$V_n = V_{3ms} = V_{ms}(V_{ms}^2 - 3) = 6V_m x^2$$

by (4.13). Since $3 \nmid P$ and $ms = 2rs \equiv 2(mod 4),$ it follows that $3 | V_{ms}$ by Lemma 4.1.2. Thus

$$(V_{ms}/V_m)((V_{ms}^2 - 3)/3) = 2x^2.$$

It can be easily seen that $(V_{ms}/V_m, (V_{ms}^2 - 3)/3) = 1$, which shows that

$$V_{ms} = 2V_m u^2 \text{ and } V_{ms}^2 - 3 = 3v^2 \tag{4.27}$$

or

$$V_{ms} = V_m u^2 \text{ and } V_{ms}^2 - 3 = 6v^2 \tag{4.28}$$

for some integers u and $v.$ Identity (4.27) is not possible by Theorem 4.2.2. Assume that (4.28) is satisfied. By Theorem 4.2.1, it follows that $s = 1.$ Therefore $V_m^2 - 3 = 6v^2.$ Using (4.11), we get $V_{2m} = V_{4r} = 6v^2 + 1.$ Assume that $r > 1.$ Then $2m = 4r = 4(4q \pm 1) = 2(2^k z) \pm 4$ for some odd integer z with $k \geq 3.$ Hence

$$V_{2m} \equiv -V_4 (mod V_{2^k})$$

by (4.4), that is,

$$6v^2 \equiv -(V_4+1)(mod V_{2^k}).$$

Since V_4 and V_{2^k} are odd by (4.14), it follows that

$$3v^2 \equiv -(V_4+1)/2(mod V_{2^k}),$$

which implies that

$$J = \left(\frac{-3(V_4+1)/2}{V_{2^k}}\right) = 1.$$

On the other hand, since $\left(\frac{-1}{V_{2^k}}\right) = -1$ by (4.9) and $\left(\frac{V_{2^k}}{3}\right) = \left(\frac{2}{3}\right) = -1$ by Lemma 4.1.1, it is seen that

$$\left(\frac{3}{V_{2^k}}\right) = (-1)^{\frac{V_{2^k}-1}{2}}\left(\frac{V_{2^k}}{3}\right) = (-1)(-1) = 1.$$

Besides, it is clear that $V_4 \equiv 7(mod 16)$ and therefore $(V_4+1)/8$ is odd. Thus

$$\left(\frac{V_{2^k}}{(V_4+1)/8}\right) = \left(\frac{-1}{(V_4+1)/8}\right)$$

by Lemma 4.2.2, and so

$$\begin{aligned} J &= \left(\frac{-3(V_4+1)/2}{V_{2^k}}\right) = \left(\frac{-1}{V_{2^k}}\right)\left(\frac{3}{V_{2^k}}\right)\left(\frac{4}{V_{2^k}}\right)\left(\frac{(V_4+1)/8}{V_{2^k}}\right) \\ &= (-1)\left(\frac{(V_4+1)/8}{V_{2^k}}\right). \end{aligned}$$

Let $y = (V_4+1)/8.$ Then,

$$\begin{aligned} J &= (-1)\left(\frac{(V_4+1)/8}{V_{2^k}}\right) = (-1)\left(\frac{y}{V_{2^k}}\right) = (-1)(-1)^{\left(\frac{y-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)}\left(\frac{V_{2^k}}{y}\right) \\ &= (-1)(-1)^{\left(\frac{y-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)}\left(\frac{-1}{y}\right) = (-1)(-1)^{\frac{y-1}{2}}(-1)^{\frac{y-1}{2}} = -1, \end{aligned}$$

which is a contradiction. Therefore $r=1$, which implies that $m=2$ and $n=6$. Since $V_2 = P^2+2$ and $V_6 = P^6+6P^4+9P^2+2$, the equation $P^4+4P^2+1=6x^2$ follows from $V_n = 6V_m x^2$. Completing the square gives $(P^2+2)^2-3=6x^2$. Then it follows that $3|(P^2+2)$ and therefore $P^2+2=3c$ for some positive integer c, which leads to the equation $3c^2-2x^2=1$. By Lemma 1 in [49], if $k\sqrt{A}+t\sqrt{B}$ is a solution of the equation $AX^2-BY^2=1$ and $r+s\sqrt{AB}$ is a solution of the equation $X^2-ABY^2=1$, then the product $(kr+Bts)\sqrt{A}+(tr+Aks)\sqrt{B}$ is a solution of the equation $AX^2-BY^2=1$. Thus, since $\sqrt{3}+\sqrt{2}$ is a solution of the equation $3c^2-2x^2=1$ and $5+2\sqrt{6}$ is a solution of the equation $c^2-6x^2=1$, the equation $3c^2-2x^2=1$ has infinitely many solutions. Therefore $m=2$ and $n=6$ is a solution.

Let $3|P$. Then, since $3|V_n$ and $2|V_n$, it follows that n is odd by Lemma 4.1.2 and $3|n$ by (4.14), respectively. Therefore $n=12q\pm 3$ for some positive integer q. Thus,

$$V_n \equiv V_{\pm 3} \equiv \pm 4P \equiv 4(mod\, 8)$$

by (4.6), that is,

$$6V_m x^2 \equiv 4(mod\, 8).$$

Also, since x is odd, it follows that $6V_m \equiv 4(mod\, 8)$. Thus V_m is even and therefore $3|m$ by (4.14). Moreover, since n is odd and $V_m|V_n$, it is seen that m is odd by (4.21). Then $V_3|V_m$ by (4.21). Besides, since P is odd, it can be seen that $4|V_3$ and therefore $4|V_m$. This shows that $8|V_n$, which is not possible by (4.7).

Finally, assume that P is even. If n is odd, then m is also odd. Hence $V_n \equiv nP(modP^2)$ and $V_m \equiv mP(modP^2)$ by Lemma 4.1.4, which implies that

$$nP \equiv 6mPx^2(modP^2),$$

or

$$n \equiv 6mx^2(modP),$$

which is not possible since P is even and n is odd. If n is even, then m is also even. Hence $V_n \equiv 2(mod\, 4)$ and $V_m \equiv 2(mod\, 4)$ by Lemma 4.1.4, which shows that

$$6V_m x^2 \equiv 0(mod\, 4),$$

or

$$V_n \equiv 0(mod\, 4),$$

which contradicts the fact that $V_n \equiv 2(mod\, 4)$. This completes the proof.

4.3. The Equations $V_n = V_r V_m x^2$, $V_n = V_m V_r$, and $U_n = U_m U_r$

In this subsection, we assume that m and r are natural numbers. It is shown that when r is even and $V_m \neq 1$, there is no integer x such that $V_n = V_r V_m x^2$. Also when $Q \equiv 1(mod\, 8)$, $V_m \neq 1$, $V_r \neq 1$ and x is even integer, the solution of the equation $V_n = V_m V_r x^2$ is found. In addition to this, when $P \not\equiv 1,5(mod\, 8)$ and $Q \equiv 3,7(mod\, 8)$, the equation $V_n = V_m V_r x^2$ is considered. Moreover, it is shown that when $P > 1$ and $Q = \pm 1$, there is no generalized Lucas number V_n such that $V_n = V_m V_r$ for $m > 1$ and $r > 1$. Finally, it is shown that there is no generalized Fibonacci number U_n such that $U_n = U_m U_r$ for $Q = \pm 1$ and $1 < r < m$.

Throughout this subsection, it is assumed that P and Q are relatively prime positive integers.

In [22], the authors showed that there is no integer x such that $L_n = L_m L_r x^2$ when m and r are natural numbers with even r. Now, the same problem is solved for generalized Lucas numbers.

Theorem 4.3.1. Let $Q \equiv 1,5(mod\, 8)$, $V_m \neq 1$, and r be odd. Then there is no integer x such that $V_n = V_m V_{2r} x^2$.

Proof: Assume that $V_n = V_{2r} V_m x^2$, $Q \equiv 1,5(mod\, 8)$ and r is odd. Firstly, assume that P is odd. Since $V_m \mid V_n$ and $V_{2r} \mid V_n$, it follows that $n = mt$ and $n = 2rs$ for some odd integers t and s by (4.21). Thus $2 \mid n$ and $n/2$ is odd. Since $n = mt$ and t is odd, it is seen that $2 \mid m$ and $m/2$ is odd. Then we can write $m = 12q + c$ with $c \in \{2, 6, 10\}$ and $q \geq 0$. Thus

$$V_m \equiv V_{12q+c} \equiv V_2, V_6, V_{10} (mod\, 8)$$

by (4.6). Since $V_2 \equiv 3(mod 8)$, $V_6 \equiv 2(mod 8)$ and $V_{10} \equiv 3(mod 8)$ by Lemma 4.1.3, it follows that

$$V_m \equiv 2,3(mod 8).$$

Similarly, it is seen that $V_n \equiv 2,3(mod 8)$ and $V_{2r} \equiv 2,3(mod 8)$. Assume that $V_{2r} \equiv 3(mod 8)$. Then $V_n = V_{2r}V_m x^2 \equiv 3V_m x^2 (mod 8)$. Moreover, $3x^2 \equiv 0,3,4(mod 8)$ and $V_m \equiv 2,3(mod 8)$, which shows that $3V_m x^2 \equiv 0,1,4,6(mod 8)$. But this contradicts the fact that $V_n \equiv 2,3(mod 8)$. Now assume that $V_{2r} \equiv 2(mod 8)$. Then $V_n = V_{2r}V_m x^2 \equiv 2V_m x^2 (mod 8)$. Since $2x^2 \equiv 0,2(mod 8)$ and $V_m \equiv 2,3(mod 8)$, it is seen that $V_n \equiv 0,4,6(mod 8)$, which contradicts the fact that $V_n \equiv 2,3(mod 8)$.

Finally, assume that P is even. Then since n is even and Q is odd, it is seen that $V_n \equiv 2(mod 4)$ by Lemma 4.1.4. Similarly, $V_m \equiv 2(mod 4)$ and $V_{2r} \equiv 2(mod 4)$. This shows that $V_n \equiv 0(mod 4)$, which contradicts the fact that $V_n \equiv 2(mod 4)$. This completes the proof.

Theorem 4.3.2. Let $Q \equiv 3,7(mod 8)$, $V_m \neq 1$, and r be odd. Then there is no integer x such that $V_n = V_m V_{2r} x^2$.

Proof: Assume that $V_n = V_{2r}V_m x^2$, $Q \equiv 3,7(mod 8)$, and r is odd. Firstly, assume that P is odd. Since $V_m | V_n$ and $V_{2r} | V_n$, it follows that $n = mt$ and $n = 2rs$ for some odd integers t and s by (4.21). Thus $2 | n$ and $n/2$ is odd. Since $n = mt$ and t is odd, it is seen that $2 | m$ and $m/2$ is odd. Then $m = 12q + c$ with $c \in \{2,6,10\}$ and $q \geq 0$. Thus

$$V_m \equiv V_{12q+c} \equiv V_2, V_6, V_{10} (mod 8)$$

by (4.6). Since $V_2 \equiv 7(mod 8)$, $V_6 \equiv 2(mod 8)$, and $V_{10} \equiv 7(mod 8)$ by Lemma 4.1.3, it follows that

$$V_m \equiv 2,7(mod 8).$$

Similarly, it is seen that $V_n \equiv 2,7(mod 8)$ and $V_{2r} \equiv 2,7(mod 8)$. Assume that $V_{2r} \equiv 7(mod 8)$. Then it follows that $V_n = V_{2r}V_m x^2 \equiv 7V_m x^2 (mod 8)$. Moreover, $7x^2 \equiv 0,4,7(mod 8)$ and $V_m \equiv 2,7(mod 8)$, which shows that $7V_m x^2 \equiv 0,1,4,6(mod 8)$. But this contradicts the fact that $V_n \equiv 2,7(mod 8)$. Now assume that $V_{2r} \equiv 2(mod 8)$. Then $V_n = V_{2r}V_m x^2 \equiv 2V_m x^2 (mod 8)$. Since $2x^2 \equiv 0,2(mod 8)$ and $V_m \equiv 2,7(mod 8)$, it is seen that $V_n \equiv 0,4,6(mod 8)$, which contradicts the fact that $V_n \equiv 2,7(mod 8)$.

Secondly, assume that P is even. Then since n is even and Q is odd, it is seen that $V_n \equiv 2(mod 4)$ by Lemma 4.1.4. Similarly, it is seen that $V_m \equiv 2(mod 4)$ and $V_{2r} \equiv 2(mod 4)$. This shows that $V_n \equiv 0(mod 4)$, which contradicts the fact that $V_n \equiv 2(mod 4)$. This completes the proof.

Theorem 4.3.3. Let $V_m \neq 1$, $k \geq 2$, and Q, r be odd. Then there is no integer x such that $V_n = V_{2^k r}V_m x^2$.

Proof: Assume that $V_n = V_{2^k r}V_m x^2$ and r is odd. Firstly, assume that P is odd. Since $V_m | V_n$ and $V_{2^k r} | V_n$, there exist two odd integers t and s such that $n = mt$ and $n = 2^k rs$ by (4.21). Thus $m = 2^k c$ for some odd positive integer c. Then $V_n \equiv 2,7(mod 8)$, $V_m \equiv 2,7(mod 8)$ and $V_{2^k r} \equiv 2,7(mod 8)$ by Lemma 4.1.3. Assume that $V_{2^k r} \equiv 2(mod 8)$. Then it follows that

$$V_n = V_{2^k r}V_m x^2 \equiv 2V_m x^2 (mod 8).$$

Moreover, since $2x^2 \equiv 0,2(mod 8)$ and $V_m \equiv 2,7(mod 8)$, it is seen that $2V_m x^2 \equiv 0,4,6(mod 8)$, which contradicts the fact that $V_n \equiv 2,7(mod 8)$. Now assume that $V_{2^k r} \equiv 7(mod 8)$. Then $V_n = V_{2^k r}V_m x^2 \equiv 7V_m x^2 (mod 8)$. Moreover, $7x^2 \equiv 0,4,7(mod 8)$ and $V_m \equiv 2,7(mod 8)$. This shows that $7V_m x^2 \equiv 0,1,4,6(mod 8)$, which contradicts the fact that $V_n \equiv 2,7(mod 8)$.

Secondly, assume that P is even. Then, since n is even and Q is odd, it is seen that $V_n \equiv 2(mod\,4)$ by Lemma 4.1.4. Similarly, it is seen that $V_m \equiv 2(mod\,4)$ and $V_{2r} \equiv 2(mod\,4)$. This shows that $V_n \equiv 0(mod\,4)$, which contradicts the fact that $V_n \equiv 2(mod\,4)$. This completes the proof.

Thus, the following theorem can be stated easily.

Theorem 4.3.4. Let r be even, Q be odd, and $V_m \neq 1$. Then there is no integer x such that $V_n = V_m V_r x^2$.

Proof: The proof follows from Theorem 4.3.1, Theorem 4.3.2, and Theorem 4.3.3.

The lemma given below is from number theory and it is used in the proof of the theorem following it.

Lemma 4.3.1. Let $a,b,c,x \in \mathbf{Z}$, $\gcd(a,b)=1$ and $ab = cx^2$. Then $a = ru^2$ and $b = sv^2$ with $rs = c$ for some positive integers u and v.

In [22], the authors showed that for $m>1$ and $r>1$, there is no even integer x such that $L_n = L_m L_r x^2$. Besides, if $Q \equiv 3,7(mod\,8)$ and x is even, then it can be seen that the equation $V_n = V_m V_r x^2$ has no solutions by (4.8). Now, the same problem is considered for $P \geq 1$ and $Q \equiv 1(mod\,8)$.

Theorem 4.3.5. Let x be an even integer and $Q \equiv 1(mod\,8)$. If $V_n = V_m V_r x^2$ with $V_m \neq 1$, $V_r \neq 1$, then $m = r = 1$, $n = 3$, and $P = 3$.

Proof: Assume that $V_n = V_m V_r x^2$, $Q \equiv 1(mod\,8)$, and x is even. If one of m and r is even, the proof follows from Theorem 4.3.4. Assume that m and r are odd. Firstly, assume that P is odd. Since x is even, it follows that $4 \mid V_n$ and therefore $3 \mid n$ by (4.14). If $3 \mid m$ or $3 \mid r$, then V_m or V_r is even by (4.14). Thus we get $8 \mid V_n$, which is impossible by (4.7). Therefore we have $3 \nmid m$ and $3 \nmid r$. Since $V_m \mid V_n$ and $V_r \mid V_n$, there exist two odd positive integers t and s such that $n = mt$ and $n = rs$ by (4.21). Then n is odd. As a result, $n = mt$, $n = rs$, $3 \mid n$, $3 \nmid m$, and $3 \nmid r$. Therefore $t = 3a$ and $s = 3b$ for some odd positive integers a and b, which shows that $n = 3ma = 3rb$, i.e., $ma = rb$. Thus, since n is odd, it follows that

$$V_m V_r x^2 = V_n = V_{3ma} = V_{ma}(V_{ma}^2 + 3Q^{ma})$$

by (4.13), which shows that

$$\frac{V_{ma}}{V_m}(V_{ma}^2 + 3Q^{ma}) = V_r x^2. \qquad (4.29)$$

Then, using (4.16), it can be seen that $\left(\frac{V_{ma}}{V_m}, V_{ma}^2 + 3Q^{ma}\right) = 1$ or 3. In both cases, by Lemma 4.3.1, we have

$$\frac{V_{ma}}{V_m} = wu_1^2 \text{ and } V_{ma}^2 + 3Q^{ma} = yu_2^2 \qquad (4.30)$$

or

$$\frac{V_{ma}}{3V_m} = wu_1^2 \text{ and } \frac{V_{ma}^2 + 3Q^{ma}}{3} = yu_2^2 \qquad (4.31)$$

with $wy = V_r$ for some positive integers $w, y, u_1,$ and u_2. Using the fact that $ma = rb$ in (4.30) and (4.31), one gets $V_{rb}^2 + 3Q^{rb} = yu_2^2$ and $V_{rb}^2 + 3Q^{rb} = 3yu_2^2$, respectively. Thus it follows that $y \mid V_{rb}^2 + 3Q^{rb}$. Since $y \mid V_r$ and $V_r \mid V_{rb}$, it is seen that $y \mid 3Q^{rb}$. Since $y \mid V_r$, one obtains $y \mid 3$ by (4.16), which shows that $y = 1$ or $y = 3$. As a result, it follows that $V_{rb}^2 + 3Q^{rb} = v^2$ or $V_{rb}^2 + 3Q^{rb} = 3v^2$ for some integer v. Assume that $V_{rb}^2 + 3Q^{rb} = v^2$. Using (4.11), one gets $V_{2rb} = v^2 - Q^{rb}$. Assume that $rb > 1$. Then $2rb = 2(4q \pm 1) = 2(2^k z) \pm 2$ for some odd positive integer z with $k \geq 2$. Hence,

$$V_{2rb} \equiv -Q^{rb-1} V_2 (mod V_{2^k})$$

or

$$V_{2rb} \equiv -Q^{rb+1} V_{-2} (mod V_{2^k})$$

by (2.17). In both cases, it is seen that

$$v^2 - Q^{rb} \equiv -Q^{rb-1}V_2(modV_{2^k})$$

by (1.1), that is,

$$v^2 \equiv -Q^{rb-1}(V_2 - Q) \equiv -Q^{rb-1}U_3(\mathrm{mod} V_{2^k}),$$

which shows that $J = \left(\frac{-Q^{rb-1}U_3}{V_{2^k}}\right) = 1$. On the other hand, $\left(\frac{U_3}{V_{2^k}}\right) = 1$ by (4.24). Moreover, $V_{2^k} \equiv 7(mod 8)$ by Lemma 4.1.3 and therefore $\left(\frac{-1}{V_{2^k}}\right) = -1$. Also since $rb-1$ is even, it is seen that $\left(\frac{Q^{rb-1}}{V_{2^k}}\right) = 1$. Thus

$$J = \left(\frac{-Q^{rb-1}U_3}{V_{2^k}}\right) = \left(\frac{-1}{V_{2^k}}\right)\left(\frac{Q^{rb-1}}{V_{2^k}}\right)\left(\frac{U_3}{V_{2^k}}\right) = -1,$$

which contradicts the fact that $J = 1$. Assume that $V_{rb}^2 + 3Q^{rb} = 3v^2$. Then $3 | V_{rb}$. This shows that $3 | P$ by Lemma 4.1.2 since rb is odd. Using (4.11), one obtains $V_{2rb} = 3v^2 - Q^{rb}$. Assume that $rb > 1$. Then it is clear that $2rb = 2(4q \pm 1) = 2(2^k z) \pm 2$ for some odd positive integer z with $k \geq 2$. Hence,

$$V_{2rb} \equiv -Q^{rb-1}V_2(modV_{2^k})$$

or

$$V_{2rb} \equiv -Q^{rb+1}V_{-2}(modV_{2^k})$$

by (2.17). In both cases, it is seen that

$$3v^2 - Q^{rb} \equiv -Q^{rb-1}V_2(modV_{2^k})$$

by (1.1). That is,

$$3v^2 \equiv -Q^{rb-1}(V_2 - Q) \equiv -Q^{rb-1}U_3(modV_{2^k}),$$

which shows that $J = \left(\frac{-3Q^{rb-1}U_3}{V_{2^k}}\right) = 1$. Besides, it is obvious that $\left(\frac{U_3}{V_{2^k}}\right) = 1$ by (4.24) and $\left(\frac{-1}{V_{2^k}}\right) = -1$ by (4.9). Since rb is odd, one gets $\left(\frac{Q^{rb-1}}{V_{2^k}}\right) = 1$. Also since

$3 | P$ and $k \geq 2$, it can be easily seen that $V_{2^k} \equiv 2Q^{\frac{2^k}{2}} (mod 3)$ by Lemma 4.1.4. Therefore

$$\left(\frac{3}{V_{2^k}}\right) = \left(\frac{V_{2^k}}{3}\right)(-1)^{\left(\frac{3-1}{2}\right)\left(\frac{V_{2^k}-1}{2}\right)} = -\left(\frac{2Q^{\frac{2^k}{2}}}{3}\right) = -\left(\frac{2}{3}\right)\left(\frac{Q^{\frac{2^k}{2}}}{3}\right) = 1.$$

Consequently,

$$J = \left(\frac{-3Q^{rb-1}U_3}{V_{2^k}}\right) = \left(\frac{-1}{V_{2^k}}\right)\left(\frac{3}{V_{2^k}}\right)\left(\frac{Q^{rb-1}}{V_{2^k}}\right)\left(\frac{U_3}{V_{2^k}}\right) = -1,$$

which contradicts the fact that $J = 1$. Therefore $rb = 1$, i.e., $r = b = 1$. This shows that $m = r = 1$ and $n = 3$. Hence, $V_3 = V_1 V_1 x^2 = (Px)^2$, i.e., $P(P^2 + 3Q) = P^2 x^2$, which implies that $P | (P^2 + 3Q)$ and therefore $P | 3Q$. Since $(P, Q) = 1$, it follows that $P | 3$. This shows that $P = 3$ since $P = V_1 = V_m \neq 1$ by the assumption.

Secondly, assume that P is even. Since x is even, it is seen that $4 | V_n$ and therefore n is odd by Lemma 4.1.4. This shows that m and r are also odd. On the other hand $V_n \equiv nPQ^{\frac{n-1}{2}} (modP^2)$, $V_m \equiv mPQ^{\frac{m-1}{2}} (modP^2)$, and $V_r \equiv rPQ^{\frac{r-1}{2}} (modP^2)$ by Lemma 4.1.4, which imply that

$$nPQ^{\frac{n-1}{2}} \equiv mrP^2 Q^{\left(\frac{m+r-2}{2}\right)} x^2 (modP^2),$$

or

$$nQ^{\frac{n-1}{2}} \equiv mrPQ^{\left(\frac{m+r-2}{2}\right)} x^2 (modP),$$

which is impossible since n and Q are odd integers. This completes the proof.

Theorem 4.3.6. Let $P \not\equiv 1,5 (mod 8)$ and $Q \equiv 3,7 (mod 8)$. Then there is no integer x such that $V_n = V_m V_r x^2$.

Proof: Assume that $V_n = V_m V_r x^2$ and $P \not\equiv 1,5 (mod 8)$. When m or r is even, the proof follows from Theorem 4.3.4. Therefore, assume that m and r are odd. Then n is also odd.

Firstly, assume that P is odd. If $3|m$ and $3|r$, then V_m and V_r are even by (4.14). Thus it follows that $4|V_n$. This is impossible by (4.8). Therefore $3\nmid m$ or $3\nmid r$. Since $4\nmid V_n$, x is an odd integer. Assume that $3\nmid m$ and $3\nmid r$. Thus $3\nmid n$. Since n,m, and r are odd, it is seen that $V_n \equiv P,5P(mod 8)$, $V_m \equiv P,5P(mod 8)$, and $V_r \equiv P,5P(mod 8)$ by (4.6). Thus one gets $V_n = V_m V_r x^2 \equiv 1,5(mod 8)$. Then $P \equiv 1,5(mod 8)$ or $5P \equiv 1,5(mod 8)$, which is impossible since $P \not\equiv 1,5(mod 8)$. Assume that $3|m$ and $3\nmid r$. Then $3|n$. If $Q \equiv 7(mod 8)$, then it follows that $V_n \equiv 6P(mod 8)$, $V_m \equiv 6P(mod 8)$, $V_r \equiv P(mod 8)$ by (4.6) and if $Q \equiv 3(mod 8)$, then $V_n \equiv 2P(mod 8), V_m \equiv 2P(mod 8)$, $V_r \equiv P,5P(mod 8)$ by (4.6). In both cases, from the equation $V_n = V_m V_r x^2$, we get that $P \equiv 1(mod 4)$, which is impossible since $P \not\equiv 1,5(mod 8)$.

Secondly, assume that P is even. Since n,m, and r are odd, it follows that $V_n \equiv nPQ^{\frac{n-1}{2}}(mod P^2)$, $V_m \equiv mPQ^{\frac{m-1}{2}}(mod P^2)$, and $V_r \equiv rPQ^{\frac{r-1}{2}}(mod P^2)$ by Lemma 4.1.4. This shows that

$$nPQ^{\frac{n-1}{2}} \equiv mrP^2 Q^{\left(\frac{m+r-2}{2}\right)} x^2 (mod P^2),$$

or

$$nQ^{\frac{n-1}{2}} \equiv mrPQ^{\left(\frac{m+r-2}{2}\right)} x^2 (mod P),$$

which is impossible since n and Q are odd. This completes the proof.

The following theorem is proved by Keskin and Demirtürk in [22] when $(P,Q) = (1,1)$.

Theorem 4.3.7. Let $P > 1$ and $Q = 1$. Then there is no generalized Lucas number V_n such that $V_n = V_m V_r$.

Proof: Assume that $V_n = V_m V_r$, $P > 1$, and $Q = 1$. If one of m and r is even, then the proof follows from Theorem 4.3.4. Therefore, assume that m and r are odd.

Firstly, assume that P is odd. Since $V_m \mid V_n$ and $V_r \mid V_n$, there exist two odd integers t and s such that $n = mt$ and $n = rs$ by (4.21). It is obvious that $t > 1$ and $s > 1$. Hence $t = 4q \pm 1$ for some $q \geq 1$ and so, $n = mt = 4mq \pm m = 2(2mq) \pm m$. Then it follows that

$$V_m V_r = V_n \equiv \pm V_m (mod V_{2m}) \tag{4.32}$$

by (4.4). Similarly, it is seen that

$$V_m V_r \equiv \pm V_r (mod V_{2r}). \tag{4.33}$$

If $3 \mid m$ and $3 \mid r$, then, since m and r are odd, it follows that $V_3 \mid V_m$ and $V_3 \mid V_r$ by (4.21). Since P is odd, it can be easily seen that $4 \mid V_3$ or $8 \mid V_n$, which is impossible by (4.7). Therefore $3 \nmid m$ or $3 \nmid r$. Assume that $3 \nmid m$ and $3 \nmid r$. Then $(V_m, V_{2m}) = (V_r, V_{2r}) = 1$ by (4.14) and (4.19). Using (4.32) and (4.33), one gets

$$V_r \equiv \pm 1 (mod V_{2m}) \tag{4.34}$$

and

$$V_m \equiv \pm 1 (mod V_{2r}), \tag{4.35}$$

respectively. Thus

$$V_{2m} \leq V_r \pm 1 \leq V_r + 1 \text{ and } V_{2r} \leq V_m \pm 1 \leq V_m + 1$$

by (4.34) and (4.35), respectively. As a result it is obtained that

$$V_{2m} + V_{2r} \leq V_m + V_r + 2. \tag{4.36}$$

Using (4.11) in (4.36), one gets $V_m^2 + V_r^2 + 2 \leq V_m + V_r$, which is impossible. Assume that $3 \mid m$ and $3 \nmid r$. Then $(V_m, V_{2m}) = 2$ and $(V_r, V_{2r}) = 1$ by (4.14) and (4.19). Hence one has

$$V_r \equiv \pm 1 (mod V_{2m}/2), \tag{4.37}$$

and

$$V_m \equiv \pm 1 (mod V_{2r}) \tag{4.38}$$

by (4.32) and (4.33), respectively. Moreover, by (4.37) and (4.38), it can be seen that

$$V_{2m} \leq 2V_r + 2 \tag{4.39}$$

and

$$V_{2r} \leq V_m + 1. \tag{4.40}$$

Then,

$$V_{2m} + V_{2r} \leq V_m + 2V_r + 3 \tag{4.41}$$

by (4.39) and (4.40). Using (4.11) in (4.41), one obtains $V_m^2 - V_m + V_r^2 - 2V_r \leq -1$, which shows that $V_m(V_m - 1) + V_r(V_r - 2) \leq -1$. But this is not possible since $V_m \geq 2$ and $V_r \geq 2$.

Secondly, assume that P is even. Since n, m, and r are odd, it follows that $V_n \equiv nP(mod P^2)$, $V_m \equiv mP(mod P^2)$, and $V_r \equiv rP(mod P^2)$ by Lemma 4.1.4. This shows that $nP \equiv mrP^2(mod P^2)$ or $n \equiv mrP(mod P)$, which is impossible since n is odd. This completes the proof.

Now, we consider the above theorem for $Q = -1$.

Theorem 4.3.8. Let $P > 1$ and $Q = -1$. Then there is no generalized Lucas number V_n such that $V_n = V_m V_r$.

Proof: Assume that $V_n = V_m V_r$ and $Q = -1$. If one of m and r is even, then the proof follows from Theorem 4.3.4. Therefore, assume that m and r are odd.

Firstly, assume that P is odd. Then, since $V_m | V_n$ and $V_r | V_n$, there exist two odd integers t and s such that $n = mt$ and $n = rs$ by (4.21). It is obvious that $t > 1$ and $s > 1$. Hence $t = 4q \pm 1$ for some $q \geq 1$, and hence $n = mt = 4mq \pm m = 2(2mq) \pm m$. Then it follows that

$$V_m V_r = V_n \equiv \pm V_m (mod V_{2m}) \tag{4.42}$$

by (1.1) and (2.17). Similarly, it is seen that

$$V_m V_r \equiv \pm V_r (mod V_{2r}). \tag{4.43}$$

If $3 | m$ and $3 | r$, then V_m and V_r are even by (4.14). This shows that $4 | V_n$, which is impossible by (4.8). Therefore $3 \nmid m$ or $3 \nmid r$. Assume that $3 \nmid m$ and $3 \nmid r$. Then $(V_m, V_{2m}) = (V_r, V_{2r}) = 1$ by (4.14) and (4.19). Hence,

$$V_r \equiv \pm 1 (mod V_{2m}) \tag{4.44}$$

by (4.42) and

$$V_m \equiv \pm 1(mod V_{2r}) \tag{4.45}$$

by (4.43). Thus one obtains

$$V_{2m} \leq V_r \pm 1 \leq V_r + 1$$

and

$$V_{2r} \leq V_m \pm 1 \leq V_m + 1$$

by (4.44) and (4.45), respectively. Then it follows that

$$V_{2m} + V_{2r} \leq V_m + V_r + 2. \tag{4.46}$$

Using (4.11) in (4.46), one gets $V_m(V_m - 1) + V_r(V_r - 1) \leq 6,$ which is impossible since $V_m \geq P \geq 3$ and $V_r \geq P \geq 3.$ Assume that $3 | m$ and $3 \nmid r$. Then $(V_m, V_{2m}) = 2$ and $(V_r, V_{2r}) = 1$ by (4.14) and (4.19). Hence

$$V_r \equiv \pm 1(mod V_{2m}/2), \tag{4.47}$$

and

$$V_m \equiv \pm 1(mod V_{2r}) \tag{4.48}$$

by (4.42) and (4.43), respectively. It can be seen that

$$V_{2m} \leq 2V_r + 2 \tag{4.49}$$

and

$$V_{2r} \leq V_m + 1 \tag{4.50}$$

by (4.47) and (4.48). Thus

$$V_{2m} + V_{2r} \leq V_m + 2V_r + 3 \tag{4.51}$$

by (4.49) and (4.50). Using (4.11) in (4.51), one obtains $V_m^2 - V_m + V_r^2 - 2V_r \leq 7.$ This shows that $V_m(V_m - 1) + V_r(V_r - 2) \leq 7,$ which is impossible since $V_m \geq P \geq 3$ and $V_r \geq P \geq 3.$

Secondly, assume that P is even. Since $n, m,$ and r are odd, $V_n \equiv \pm nP(mod P^2),$ $V_m \equiv \pm mP(mod P^2)$, and $V_r \equiv \pm rP(mod P^2)$ by Lemma 4.1.4. This shows that $nP \equiv \pm mrP^2(mod P^2).$ This implies that $n \equiv \pm mrP(mod P),$ which is impossible since n is odd. This completes the proof.

The following lemma is given without proof since it is easy.

Lemma 4.3.2. If $Q = \pm 1$ and $0 < r < n,$ then $V_n > 2U_r$.

In [13], Farrokhi showed that the equation $F_n = F_m F_r$ has no solutions for $m > 2$ and $r > 2.$ Now a similar result for generalized Fibonacci numbers when $P > 1$ and $Q = \pm 1$ is given.

Theorem 4.3.9. Let $P > 1,$ $Q = \pm 1$ and $m > r > 1.$ Then there is no generalized Fibonacci number U_n such that $U_n = U_m U_r.$

Proof: Assume that $U_n = U_m U_r,$ $Q = \pm 1$ and $m > r > 1.$ Then since $U_m | U_n$ and $U_r | U_n,$ there exist two positive integers t and s such that $n = mt$ and $n = rs$ by (4.20).

Firstly, assume that t is even, i.e., $t = 2a$ for some positive integer $a.$ Then $n = mt = 2ma.$ Thus it follows that $U_m U_r = U_n = U_{2ma} = U_{ma} V_{ma}$ by (4.10). This shows that $(U_{ma}/U_m) V_{ma} = U_r$ by (4.20). Therefore $V_{ma} | U_r.$ By (4.22), one obtains $r = 2mac = nc$ for some natural number $c.$ This shows that $n | r.$ Since $r | n,$ it follows that $n = r.$ Therefore $U_m = 1,$ which is impossible since $m > 1$ and $P > 1.$

Secondly, assume that t is odd. It is obvious that $t > 1.$ Then one can write $t = 4q \pm 1$ with $q \geq 1.$ Therefore $n = mt = 2(2mq) \pm m.$ Thus it follows that

$$U_n = U_{2(2mq) \pm m} \equiv U_{\pm m} (mod U_{2m}),$$

by (2.18). Using (1.1), one gets

$$U_m U_r \equiv \pm U_m (mod U_{2m}) . \tag{4.52}$$

Since $U_{2m} = U_m V_m$ by (4.10), it follows that

$$U_r \equiv \pm 1 (mod V_m).$$

Hence $V_m \leq U_r \pm 1 \leq U_r + 1$. Moreover, since $m > r > 1$, it follows that $V_m > 2U_r$ by Lemma 4.3.2. Thus it is seen that $U_r + 1 \geq V_m > 2U_r$, which is impossible. This completes the proof.

It is well known that the greatest common divisor of U_m and U_n is again a generalized Fibonacci number by (4.17). But, the least common multiple of U_m and U_n may not be a generalized Fibonacci number. This follows from the following theorem. Since the proof of the theorem is similar to that of Theorem 4.3.9, we omit it.

Theorem 4.3.10. Let $Q = \pm 1$, $1 < m < n$, and $P > 1$. Then $[U_m, U_n]$, the least common multiple of U_m and U_n, is a generalized Fibonacci number if and only if $U_m \mid U_n$.

CHAPTER 5. CONCLUSIONS AND RECOMMENDATIONS

The second chapter of this thesis is accepted for publication in “Hacettepe Journal of Mathematics and Statistics” [47]. The third chapter is published in “Journal of Integer Sequences” [23].

Moreover, in this thesis, firstly, we focused on the equations $F_n = wF_m x^2$ and $L_n = wL_m x^2$ with $w \in \{1,2,3,6\}$. Then, we have considered corresponding equations for generalized Fibonacci and Lucas numbers, in particular for some even integer P. But, finding solutions of the equation $V_n = wx^2, w \in \{1,2,3,6\}$, is still an open problem when P is even. If the solutions of the equation $V_n = wx^2$ were known when P is even, then the equation $U_n = wU_m x^2$ could be solved when P is even. Apart from these, we solved the equation $V_n = kx^2$ when P is odd and $k \mid P$. Similarly, the solutions of the equation $U_n = kx^2$ can be investigated when $k \mid P$.

REFERENCES

[1] ALFRED, B. U., On square Lucas numbers, Fibonacci Quart., 2, 1, 11–12, 1964.

[2] ANTONIADIS, J. A., Generalized Fibonacci numbers and some Diophantine equations, Fibonacci Quart., 23, 3, 199–213, 1985.

[3] ANTONIADIS, J. A., Fibonacci and Lucas numbers of the form $3z^2 \pm 1$, Fibonacci Quart., 23, 4, 300–307, 1985.

[4] BURTON, D. M., Elementary Number Theory, McGraw -Hill Comp. Inc., 1998.

[5] CARLITZ, L., A note on Fibonacci numbers, Fibonacci Quart., 1, 1, 15–28, 1964.

[6] COHN, J. H. E., On square Fibonacci numbers, etc., J. London Math. Soc., 39, 1, 537–540, 1964.

[7] COHN, J. H. E., Square Fibonacci numbers, etc., Fibonacci Quart., 2, 2, 109–113, 1964.

[8] COHN, J. H. E., Lucas and Fibonacci numbers and some Diophantine equations, Proc. Glasgow Math. Assoc., 7, 1, 24–28, 1965.

[9] COHN, J. H. E., Eight Diophantine equation, Proc. London Math. Soc., 16, 1, 153–166, 1966.

[10] COHN, J. H. E., Five Diophantine equation, Math. Scand., 21, 1, 61–70, 1967.

[11] COHN, J. H. E., Squares in some recurrent sequences, Pacific J. Math., 41, 3, 631–646, 1972.

[12] COHN, J. H. E., Twelve Diophantine equations, Arch. Math., 65, 130–133, 1995.

[13] FARROKHI D. G., M., Some remarks on the equation $F_n = kF_m$ in Fibonacci numbers, Journal of Integer Sequences, 10, article no: 07.5.7, 1–9, 2007.

[14] HE, T.-X., SHIUE, P. J. S., On Sequences of Numbers and Polynomials Defined By Linear Recurrence Relations of Order 2, International Journal of Mathematics and Mathematical Sciences, Volume 2009, 21 page, 2009.

[15] HILTON, P., PEDERSEN, J., SOMER, L., On Lucasian Numbers, Fibonacci Quart., 35, 1, 43–47, 1997.

[16] HILTON, P., PEDERSEN, J., On Generalised Fibonaccian and Lucasian Numbers, The Mathematical Gazette, 90 (518), 215–222, 2006.

[17] HORADAM, A. F., Basic properties of a certain generalized sequence of numbers, Fibonacci Quart., 3, 3, 161–176, 1965.

[18] JONES, J. P., Representation of Solutions of Pell equations Using Lucas Sequences, Acta Academia Pead. Agr., Sectio Mathematicae, 30, 75–86, 2003.

[19] KAGAWA, T., TERAI, N., Squares in Lucas sequences and some Diophantine equations, Manuscripta Math., 96, 2, 195–202, 1998.

[20] KALMAN, D., MENA, R., The Fibonacci Numbers-Exposed, Mathematics Magazine, 76, 3, 167–181, 2003.

[21] KESKİN, R., DEMİRTÜRK, B., Some New Fibonacci and Lucas Identities by Matrix Methods, International Journal of Mathematical Education in Science and Technology, 41,3,379–387, 2009.

[22] KESKİN, R., DEMİRTÜRK, B., Fibonacci and Lucas congruences and their applications, Acta Math. Sin. (Engl. Ser.), 27, 4, 725–736, 2011.

[23] KESKİN, R., YOSMA, Z., On Fibonacci and Lucas numbers of the form cx^2, Journal of Integer Sequences, 14, article no: 11.9.3, 1–12, 2011.

[24] KESKİN, R., Solutions of Some Quadratic Diophantine Equations, Computers and Mathematics With Applications, 60, 8, 2225–2230, 2010.

[25] KING, C. H., Some Properties of the Fibonacci Numbers, Master's Thesis, San Jose State College, June 1960.

[26] KOSHY, T., Fibonacci and Lucas Numbers With Applications, John Wiley and Sons, Proc., New York-Toronto, 2001.

[27] LJUNGGREN, W., Zur theorie der gleichung $x^2 + 1 = Dy^4$, Det Norske Vid. Akad. Avh. I, 5, 333–341, 1942.

[28] LUCAS, E., Theorie des Fonctions Numeriques Simplement Periodiques, American Journal of Mathematics, 1, 2, 184–196, 1878.

[29] MCDANIEL, W. L., The g.c.d. in Lucas sequences and Lehmer number sequences, Fibonacci Quart., 29, 1, 24–30, 1991.

[30] MCDANIEL, W. L., Diophantine Representation of Lucas sequences, Fibonacci Quart., 33, 1, 59–63, 1995.

[31] MIGNOTTE, M., PETHO, A., Sur les carres dans certanies suites de Lucas, Journal de Theorie des nombers de Bordeaux, 5, 2, 333–341, 1993.

[32] MOSER, L., CARLITZ, L., Advanced problem H-2, Fibonacci Quart. 1,1, 46, 1963.

[33] MUSKAT, J. B., Generalized Fibonacci and Lucas sequences and rootfinding methods, Mathematics of Computation, 61, 203, 365–372, 1993.

[34] NAKAMULA, K., PETHO, A., Squares in binary recurrence sequences, Number Theory: Diophantine, computational and algebraic aspects; proceedings of the international conference, de Gruyter, 409–421, 1998.

[35] NIVEN, I., ZUCKERMAN, H. S., MONTGOMERY, H. L., An Introduction to The Theory of Numbers, John Wiley & Sons, Inc., Canada, 1991.

[36] OGILVY, S. C., Tommorow's math, unsolved problems for the amateur, Oxford University, 1962.

[37] RABINOWITZ, S., Algorithmic Manipulation of Fibonacci Identities, Applications of Fibonacci Numbers, 6, 389–408, 1996.

[38] RIBENBOIM, P., Square classes of Fibonacci and Lucas numbers, Portugal. Math. 46, 2, 159–175, 1989.

[39] RIBENBOIM, P., MCDANIEL, W. L., The square terms in Lucas sequences, Journal of Number Theory, 58, 1, 104–123, 1996.

[40] RIBENBOIM, P., MCDANIEL, W. L., Squares in Lucas sequences having an even first parameter, Colloquium Mathematicum, 78, 1/2, 29–34, 1998.

[41] RIBENBOIM, P., My numbers, My Friends, Springer-Verlag New York, Inc., 2000.

[42] RIBENBOIM, P., MCDANIEL, W. L., On Lucas sequence terms of the form kx^2, Number Theory: proceedings of the Turku symposium on Number Theory in memory of Kustaa Inkeri (Turku, 1999), de Gruyter, Berlin, 293–303, 2001.

[43] ROBBINS, N., On Fibonacci numbers of the form px^2, where p is prime,

Fibonacci Quart., 21, 4, 266–271, 1983.

[44] ROBBINS, N., Fibonacci numbers of the form cx^2, where $1 \leq c \leq 1000$, Fibonacci Quart., 28, 4, 306–315, 1990.

[45] ROBBINS, N., Lucas numbers of the form px^2, where p is prime, Internat. J. Math. Math. Sci., 14, 4, 697–703, 1991.

[46] ROLLET, A. P., Advanced problem 5080, Amer. Math. Monthly, 70, 216, 1963.

[47] ŞİAR, Z., KESKİN, R., Some new identities concerning generalized Fibonacci and Lucas numbers, Hacettepe Journal of Mathematics and Statistics (accepted).

[48] VAJDA, S., Fibonacci and Lucas Numbers and The Golden Section, Ellis Horwood Limited Publ., England, 1989.

[49] WALKER, D. T., On the Diophantine Equation $mX^2 - nY^2 = \pm 1$, The American Mathematical Monthly, 74, 504–513, 1967.

[50] WYLER, O., In the Fibonacci series $F_1 = 1$, $F_2 = 1, F_{n+1} = F_n + F_{n-1}$ the first, second and twelfth terms are squares, Amer. Math. Monthly, 71, 221–222, 1964.

[51] ZHIWEI, S., Singlefold Diophantine Representation of the Sequence $u_0 = 0, u_1 = 1$ and $u_{n+2} = mu_{n+1} + u_n$, Pure and Applied Logic, Beijing Univ. Press, Beijing, 97–101, 1992.

[52] ZHOU, C., A general conclusion on Lucas numbers of the form px^2, where p is prime, Fibonacci Quart., 37, 1, 39–45, 1999.

CURRICULUM VITAE

Zafer Şiar, was born in Elazığ, in 14.03.1984. He completed primary, secondary and high school in Istanbul. In 2001, he graduated from Sultanbeyli Trade Vocational School. In 2001, he entered Sakarya University, Department of Mathematics, and graduated in 2005. Between 2005-2008 years, at Sakarya University, he completed his master's degree in mathematics. Moreover, he worked as a mathematics teacher during 2006-2009 years in Sınav Dershanesi. In 2008, at Sakarya University, he began doctor of philosophy in mathematics. Zafer Şiar started working as a research assistant at Bilecik Seyh Edebali University in 2012 and still works there.